Solutions of the exercises of the book “Mechanics of Continuous Media : an Introduction”

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Preface

We taught mechanics of continuous media for two decades at EPFL (Swiss Institute of Technology in Lausanne) and we used a lot of various exercises to illustrate the subject.

We have decided to make available these solutions in order to help as much as possible the understanding of the lecture contents and mastering of the concepts. The reader will note that some solutions are sometimes simple to set up. Others are more elaborate, need longer development and are more difficult to tackle.

We will refer to the equations of the monograph “Mechanics of Continuous Media : an Introduction” published by EPFL Press, by prefixing their numbers with the bold character **B** for **B**ook. The present book of solutions possesses its own numbering.

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Table des matières

Chapitre 1

Cartesian tensors

Solution 1.1

Through the relations (**B**1.52) and (**B**1.15), one has

$$\delta'_{pq} = c_{pi} c_{qj} \delta_{ij} = c_{pj} c_{qj} = \delta_{pq} \ .$$

Solution 1.2

By the Kronecker symbol properties, one obtains

$$\begin{aligned} \delta_{ij}\delta_{ik}\delta_{jk} &= \delta_{11}\delta_{11}\delta_{11} + \delta_{22}\delta_{22}\delta_{22} + \delta_{33}\delta_{33}\delta_{33} = \\ &= 1 + 1 + 1 \\ &= 3 \ . \end{aligned}$$

One can also demonstrate the relation as follows

$$\delta_{ij}\delta_{ik}\delta_{jk} = \delta_{ij}\delta_{ij} = \delta_{ii} = 3 \ .$$

Solution 1.3

For the first relation, one has

$$\begin{aligned} \varepsilon_{ijk} u_i u_j = \varepsilon_{123} u_1 u_2 + \varepsilon_{231} u_2 u_3 + \varepsilon_{312} u_3 u_1 \\ + \varepsilon_{132} u_1 u_3 + \varepsilon_{321} u_3 u_2 + \varepsilon_{213} u_2 u_1 = 0 \ . \end{aligned}$$

For the second relation, as

$$\begin{cases} \delta_{ij} \neq 0 & \text{if } i = j \\ \varepsilon_{ijk} = 0 & \text{if } i = j, \end{cases}$$

it follows that

$$\delta_{ij}\varepsilon_{ijk} = 0 \ .$$

Solution 1.4

It is asked to demonstrate the following relation

$$\boldsymbol{t} \times (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{t} \cdot \boldsymbol{v})\boldsymbol{u} - (\boldsymbol{t} \cdot \boldsymbol{u})\boldsymbol{v} \ . \tag{1.1}$$

By definition of vector product (**B**1.31), one gets

$$(\boldsymbol{u} \times \boldsymbol{v})_k = \varepsilon_{klm} u_l v_m \ .$$

Therefore the left hand side of (1.1) yields

$$(\boldsymbol{t} \times (\boldsymbol{u} \times \boldsymbol{v}))_i = \varepsilon_{ijk} t_j (\varepsilon_{klm} u_l v_m) = \varepsilon_{ijk} \varepsilon_{klm} t_j u_l v_m \ . \tag{1.2}$$

By (**B**1.30), we know that

$$\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \ .$$

Combining (1.2) and (**B**1.30), one obtains successively

$$\begin{aligned} \varepsilon_{ijk} \varepsilon_{klm} t_j u_l v_m &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) t_j u_l v_m \\ &= \delta_{il} \delta_{jm} t_j u_l v_m - \delta_{im} \delta_{jl} t_j u_l v_m \\ &= u_i t_m v_m - v_i t_j u_j \\ &= (\boldsymbol{t} \cdot \boldsymbol{v})\boldsymbol{u} - (\boldsymbol{t} \cdot \boldsymbol{u})\boldsymbol{v} \ . \end{aligned}$$

Solution 1.5

Note that

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \varepsilon_{ijk} a_j b_k \quad \text{and} \quad (\boldsymbol{c} \times \boldsymbol{d})_l = \varepsilon_{lmn} c_m d_n \ .$$

Then one writes

$$((\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}))_r = \varepsilon_{ril} (\varepsilon_{ijk} a_j b_k)(\varepsilon_{lmn} c_m d_n) \ .$$

Let us note that $\varepsilon_{ril} = \varepsilon_{ilr} = \varepsilon_{lri}$. Consequently

$$\varepsilon_{lri} (\varepsilon_{ijk} a_j b_k)(\varepsilon_{lmn} c_m d_n) = \varepsilon_{lri} \varepsilon_{lmn} \varepsilon_{ijk} a_j b_k c_m d_n \ .$$

Using identity (**B**1.30), $\varepsilon_{lri} \varepsilon_{lmn} = \delta_{rm} \delta_{in} - \delta_{rn} \delta_{im}$, one obtains

$$\begin{aligned} \varepsilon_{ril} (\varepsilon_{ijk} a_j b_k)(\varepsilon_{lmn} c_m d_n) &= (\delta_{rm} \delta_{in} - \delta_{rn} \delta_{im}) \varepsilon_{ijk} a_j b_k c_m d_n \\ &= \varepsilon_{ijk} \delta_{rm} \delta_{in} a_j b_k c_m d_n - \varepsilon_{ijk} \delta_{rn} \delta_{im} a_j b_k c_m d_n \\ &= \varepsilon_{ijk} a_j b_k c_r d_i - \varepsilon_{ijk} a_j b_k c_i d_r \\ &= \varepsilon_{jki} b_k d_i a_j c_r - \varepsilon_{jki} b_k c_i a_j d_r \\ &= (\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{d}))\, c_r - (\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}))\, d_r \ . \end{aligned}$$

Finally

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{d}))\, \boldsymbol{c} - (\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}))\, \boldsymbol{d} \ .$$

Solution 1.6

Identity (**B**1.230)

Expressing it in index notation one has

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{a}\times\boldsymbol{b}) &= \frac{\partial}{\partial x_j}(\varepsilon_{jkl}a_k b_l)\\ &= \varepsilon_{jkl}a_k\frac{\partial b_l}{\partial x_j}+\varepsilon_{jkl}b_l\frac{\partial a_k}{\partial x_j}\\ &= -\varepsilon_{kjl}a_k\frac{\partial b_l}{\partial x_j}+\varepsilon_{ljk}b_l\frac{\partial a_k}{\partial x_j}\\ &= -\boldsymbol{a}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})+(\boldsymbol{\nabla}\times\boldsymbol{a})\cdot\boldsymbol{b}\ .\end{aligned}$$

Identity (**B**1.231)

By definition, the expression $(\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{b}$ with index notation is written as

$$((\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{b})_i = a_j\frac{\partial b_i}{\partial x_j}\ .$$

With the curl definition (**B**1.177), one has

$$(\nabla\times\boldsymbol{b})_i = (\mathbf{curl}\,\boldsymbol{b})_i = \varepsilon_{ijk}\frac{\partial b_k}{\partial x_j}\ .$$

As a consequence,

$$(\boldsymbol{a}\times(\nabla\times\boldsymbol{b}))_m = \varepsilon_{mni}\varepsilon_{ijk}a_n\frac{\partial b_k}{\partial x_j}\ .$$

With (**B**1.30), one has

$$\varepsilon_{mni}\varepsilon_{ijk} = \varepsilon_{imn}\varepsilon_{ijk} = \delta_{mj}\delta_{nk}-\delta_{mk}\delta_{nj}\ .$$

The third term of the right hand side of (**B**1.231) becomes

$$\begin{aligned}(\boldsymbol{a}\times(\nabla\times\boldsymbol{b}))_m &= (\delta_{mj}\delta_{nk}-\delta_{mk}\delta_{nj})a_n\frac{\partial b_k}{\partial x_j}\\ &= \delta_{mj}\delta_{nk}a_n\frac{\partial b_k}{\partial x_j}-\delta_{mk}\delta_{nj}a_n\frac{\partial b_k}{\partial x_j}\\ &= a_k\frac{\partial b_k}{\partial x_m}-a_j\frac{\partial b_m}{\partial x_j}\ .\end{aligned}$$

Accordingly, one finds

$$\begin{aligned}a_j\frac{\partial b_i}{\partial x_j}+b_j\frac{\partial a_i}{\partial x_j}+a_n\frac{\partial b_n}{\partial x_i}-a_j\frac{\partial b_i}{\partial x_j}+b_n\frac{\partial a_n}{\partial x_i}-b_j\frac{\partial a_i}{\partial x_j} &= a_n\frac{\partial b_n}{\partial x_i}+b_n\frac{\partial a_n}{\partial x_i}\\ &= \frac{\partial(a_n b_n)}{\partial x_i} = (\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{b}))_i\ .\end{aligned}$$

Identity (**B**1.232)

The right hand side of (**B**1.232) is written as

$$b_j \frac{\partial a_i}{\partial x_j} - a_j \frac{\partial b_i}{\partial x_j} + a_i \frac{\partial b_j}{\partial x_j} - b_i \frac{\partial a_j}{\partial x_j} = \frac{\partial (a_i b_j)}{\partial x_j} - \frac{\partial (a_j b_i)}{\partial x_j} \ .$$

The left hand side gives

$$\varepsilon_{mni} \frac{\partial (\varepsilon_{ijk} a_j b_k)}{\partial x_n} = \varepsilon_{mni} \varepsilon_{ijk} \frac{\partial (a_j b_k)}{\partial x_n} \ . \tag{1.3}$$

By (**B**1.30), one has

$$\varepsilon_{mni}\varepsilon_{ijk} = \varepsilon_{imn}\varepsilon_{ijk} = \delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj} \ .$$

Relation (1.3) becomes

$$(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}) \frac{\partial (a_j b_k)}{\partial x_n} = \frac{\partial (a_m b_n)}{\partial x_n} - \frac{\partial (a_n b_m)}{\partial x_n} \ .$$

Identity (**B**1.233)

We successively obtain

$$\begin{aligned} \frac{\partial}{\partial x_j}(a_i b_j) &= a_i \frac{\partial b_j}{\partial x_j} + b_j \frac{\partial a_i}{\partial x_j} \\ &= \boldsymbol{a} \operatorname{div} \boldsymbol{b} + (\boldsymbol{\nabla} \boldsymbol{a}) \boldsymbol{b} \ . \end{aligned}$$

Solution 1.7

Identity (**B**1.234)

One writes successively

$$\begin{aligned} (\mathbf{curl}(\Phi \boldsymbol{a}))_i &= \varepsilon_{ijk} \frac{\partial (\Phi a_k)}{\partial x_j} \\ &= \varepsilon_{ijk} \frac{\partial \Phi}{\partial x_j} a_k + \varepsilon_{ijk} \Phi \frac{\partial a_k}{\partial x_j} \\ &= \boldsymbol{\nabla} \Phi \times \boldsymbol{a} + \Phi \, \mathbf{curl} \, \boldsymbol{a} \\ &= -\boldsymbol{a} \times \boldsymbol{\nabla} \Phi + \Phi \, \mathbf{curl} \, \boldsymbol{a} \ . \end{aligned}$$

Identity (**B**1.235)

One has

$$\begin{aligned} (\boldsymbol{\nabla}(\Phi \boldsymbol{a})))_i &= \frac{\partial \Phi a_i}{\partial x_j} \\ &= \Phi \frac{\partial a_i}{\partial x_j} + a_i \frac{\partial \Phi}{\partial x_j} \\ &= \Phi \boldsymbol{\nabla} \boldsymbol{a} + \boldsymbol{a} \otimes \boldsymbol{\nabla} \Phi \ . \end{aligned}$$

Identity (**B**1.236)

The j component of gradient of Φ yields

$$(\boldsymbol{\nabla}\Phi)_j = \frac{\partial \Phi}{\partial x_j} \ .$$

Therefore

$$\left(\left(\nabla^2\left(\boldsymbol{\nabla}\Phi\right)\right)_j = \frac{\partial^2}{\partial x_i \partial x_i}\left(\frac{\partial \Phi}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial^2 \Phi}{\partial x_i \partial x_i}\right) = \left(\boldsymbol{\nabla}(\nabla^2\Phi)\right)_j \ .$$

Thus

$$\nabla^2\left(\boldsymbol{\nabla}\Phi\right) = \boldsymbol{\nabla}\left(\nabla^2\Phi\right) \ .$$

Identity (**B**1.237)

The l component of the vector corresponding to the left hand side is such that

$$\begin{aligned}\left(\boldsymbol{\nabla}\times\left(\nabla^2\boldsymbol{a}\right)\right)_l &= \left(\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial^2 a_k}{\partial x_m \partial x_m}\right)\right)_l = \left(\varepsilon_{ijk}\frac{\partial^2}{\partial x_m \partial x_m}\left(\frac{\partial a_k}{\partial x_j}\right)\right)_l \\ &= \left(\frac{\partial^2}{\partial x_m \partial x_m}\left(\varepsilon_{ijk}\frac{\partial a_k}{\partial x_j}\right)\right)_l = \left(\nabla^2\left(\boldsymbol{\nabla}\times\boldsymbol{a}\right)\right)_l \ .\end{aligned}$$

Identity (**B**1.238)

First indexed equality

$$(\Delta\boldsymbol{a})_k = \frac{\partial^2 a_k}{\partial x_m \partial x_m} = \frac{\partial}{\partial x_m}\left(\frac{\partial a_k}{\partial x_m}\right) = \left(\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\boldsymbol{a}\right)\right)_k \ .$$

Second equality

$$\begin{aligned}(\boldsymbol{\nabla}\times\boldsymbol{a})_i &= \varepsilon_{ijk}\frac{\partial a_k}{\partial x_j} \\ \left(\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\boldsymbol{a}\right)\right)_l &= \varepsilon_{lmi}\frac{\partial}{\partial x_m}\left(\varepsilon_{ijk}\frac{\partial a_k}{\partial x_j}\right) = \varepsilon_{lmi}\varepsilon_{ijk}\frac{\partial^2 a_k}{\partial x_m \partial x_j} \ .\end{aligned}$$

Note that by (**B**1.30), one has

$$\varepsilon_{lmi}\varepsilon_{ijk} = \varepsilon_{ilm}\varepsilon_{ijk} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj} \ .$$

Therefore, one obtains

$$\begin{aligned}\varepsilon_{lmi}\varepsilon_{ijk}\frac{\partial^2 a_k}{\partial x_m \partial x_j} &= \left(\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}\right)\frac{\partial^2 a_k}{\partial x_m \partial x_j} \\ &= \delta_{lj}\delta_{mk}\frac{\partial^2 a_k}{\partial x_m \partial x_j} - \delta_{lk}\delta_{mj}\frac{\partial^2 a_k}{\partial x_m \partial x_j}\end{aligned}$$

and then

$$\begin{aligned}(\nabla\times(\nabla\times\boldsymbol{a}))_l &= \frac{\partial^2 a_m}{\partial x_m\partial x_l}-\frac{\partial^2 a_l}{\partial x_j\partial x_j}\\ &= \frac{\partial}{\partial x_l}\left(\frac{\partial a_m}{\partial x_m}\right)-\frac{\partial^2 a_l}{\partial x_j\partial x_j}=\frac{\partial}{\partial x_l}\left(\boldsymbol{\nabla}\cdot\boldsymbol{a}\right)-\Delta a_l\\ &= \left(\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{a})-\nabla^2\boldsymbol{a}\right)_l\ .\end{aligned}$$

This last relation is valid for each component of the vector function $\boldsymbol{a}$

$$\Delta\boldsymbol{a}=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{a})-\mathbf{curl}\ \mathbf{curl}\ \boldsymbol{a}\ .$$

Solution 1.8

Identity (**B**1.239)

The i component of the vector corresponding to the left hand side is such that

$$\begin{aligned}(\nabla\left(\boldsymbol{a}\cdot\boldsymbol{x}\right))_i &= \left(\frac{\partial a_j}{\partial x_i}x_j+a_j\delta_{ij}\right)\\ &= \left(\frac{\partial a_j}{\partial x_i}x_j+a_i\right)=\left(\boldsymbol{a}+\left(\nabla\boldsymbol{a}\right)^T\boldsymbol{x}\right)_i\ .\end{aligned}$$

Thus

$$\nabla\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)=\boldsymbol{a}+\left(\nabla\boldsymbol{a}\right)^T\boldsymbol{x}\ .$$

Identity (**B**1.240)

The Laplacian is written

$$\nabla^2\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)=\frac{\partial^2(a_ix_i)}{\partial x_j\partial x_j}=\frac{\partial}{\partial x_j}\left(\frac{\partial(a_ix_i)}{\partial x_j}\right)\ .$$

Carrying out the algebra, one has

$$\begin{aligned}\frac{\partial}{\partial x_j}\left(\frac{\partial(a_ix_i)}{\partial x_j}\right) &= \frac{\partial}{\partial x_j}\left(\frac{\partial a_i}{\partial x_j}x_i+a_i\frac{\partial x_i}{\partial x_j}\right)=\frac{\partial}{\partial x_j}\left(\frac{\partial a_i}{\partial x_j}x_i+a_i\delta_{ij}\right)\\ &= \frac{\partial^2 a_i}{\partial x_j\partial x_j}x_i+\frac{\partial a_i}{\partial x_j}\frac{\partial x_i}{\partial x_j}+\delta_{ij}\frac{\partial a_i}{\partial x_j}\\ &= \frac{\partial^2 a_i}{\partial x_j\partial x_j}x_i+\frac{\partial a_i}{\partial x_j}\delta_{ij}+\delta_{ij}\frac{\partial a_i}{\partial x_j}=\frac{\partial^2 a_i}{\partial x_j\partial x_j}x_i+2\frac{\partial a_i}{\partial x_j}\delta_{ij}\\ &= (\nabla^2\boldsymbol{a})_i+2(\mathrm{div}\ \boldsymbol{a})_i\ .\end{aligned}$$

Thus

$$\nabla^2\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)=2\,\mathrm{div}\ \boldsymbol{a}+\boldsymbol{x}\cdot\left(\nabla^2\boldsymbol{a}\right)\ .$$

Identity ($\mathbf{B}$1.241)

The Laplacian is written

$$\begin{aligned}\left(\nabla^2\left(\Phi \boldsymbol{x}\right)\right)_i &= \frac{\partial^2(\Phi x_i)}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial(\Phi x_i)}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial x_j} x_i + \Phi \frac{\partial x_i}{\partial x_j}\right) \\ &= \frac{\partial^2 \Phi}{\partial x_j \partial x_j} x_i + \frac{\partial \Phi}{\partial x_j}\delta_{ij} + \frac{\partial \Phi}{\partial x_j}\delta_{ij} = \frac{\partial^2 \Phi}{\partial x_j \partial x_j} x_i + 2\frac{\partial \Phi}{\partial x_i} \ .\end{aligned}$$

Thus

$$\begin{aligned}\left(\nabla^2\left(\Phi \boldsymbol{x}\right)\right)_i &= (\boldsymbol{x}\nabla^2\Phi)_i + 2(\nabla\phi)_i \\ \nabla^2\left(\Phi \boldsymbol{x}\right) &= 2\nabla\Phi + \boldsymbol{x}\nabla^2\Phi \ .\end{aligned}$$

Solution 1.9

One writes with ($\mathbf{B}$1.52)

$$\begin{aligned}L'_{ij} &= \boldsymbol{e}'_i \cdot \boldsymbol{L}\boldsymbol{e}'_j = \\ &= (c_{ik}\boldsymbol{e}_k) \cdot \boldsymbol{L}(c_{jl}\boldsymbol{e}_l) = \\ &= c_{ik}c_{jl}\boldsymbol{e}_k \cdot \boldsymbol{L}\boldsymbol{e}_l = \\ &= c_{ik}c_{jl}L_{kl} \ .\end{aligned}$$

Moreover, by ($\mathbf{B}$1.15)

$$c_{ik}c_{il} = \delta_{kl} \ .$$

One has

$$\begin{aligned}tr(\boldsymbol{L}') &= L'_{ii} \\ &= c_{ik}c_{il}L_{kl} \\ &= \delta_{kl}L_{kl} \\ &= L_{kk} \\ &= tr(\boldsymbol{L}) \ .\end{aligned}$$

Solution (1.10)

Identity ($\mathbf{B}$1.69)

One has successively

$$\begin{aligned}\boldsymbol{u} \cdot \boldsymbol{L}^T\boldsymbol{v} &= u_i\left(\boldsymbol{L}^T\right)_{im} v_m = u_i L_{mi} v_m = L_{mi} u_i v_m = (\boldsymbol{L}\boldsymbol{u} \cdot \boldsymbol{v}) \\ \boldsymbol{u} \cdot \boldsymbol{L}^T\boldsymbol{v} &= u_i\left(\boldsymbol{L}^T\right)_{im} v_m = u_i L_{mi} v_m = v_m L_{mi} u_i = (\boldsymbol{v} \cdot \boldsymbol{L}\boldsymbol{u}) \ .\end{aligned}$$

Identity ($\mathbf{B}$1.71)

By definition of the scalar product of two tensors, one obtains

$$\left(\left(\boldsymbol{a} \otimes \boldsymbol{b}\right)\boldsymbol{L}\right)_{ij} = \left(\boldsymbol{a} \otimes \boldsymbol{b}\right)_{ik} L_{kj} = a_i b_k L_{kj} = a_i\left(\boldsymbol{L}^T\right)_{jk} b_k = \left(\boldsymbol{a} \otimes \boldsymbol{L}^T\boldsymbol{b}\right)_{ij} \ .$$

Solution 1.11

$$I_1(\boldsymbol{L}) = L_{11} + L_{22} + L_{33} = tr\boldsymbol{L} \ .$$

$$\begin{aligned}
I_2(\boldsymbol{L}) &= L_{11}L_{22} - L_{21}L_{12} + L_{22}L_{33} - L_{23}L_{32} + L_{11}L_{33} - L_{13}L_{31} = \\
&= (L_{11}L_{22} + L_{22}L_{33} + L_{11}L_{33}) - (L_{21}L_{12} + L_{23}L_{32} + L_{13}L_{31}) = \\
&= \frac{1}{2}(L_{11} + L_{22} + L_{33})^2 - \frac{1}{2}(L_{11}^2 + L_{22}^2 + L_{33}^2)] \\
&- (L_{21}L_{12} + L_{23}L_{32} + L_{13}L_{31}) = \\
&= \frac{1}{2}(L_{11} + L_{22} + L_{33})^2 \\
&- \frac{1}{2}(L_{11}^2 + L_{22}^2 + L_{33}^2 + 2L_{21}L_{12} + 2L_{23}L_{32} + 2L_{13}L_{31}) = \\
&= \frac{1}{2}(tr\boldsymbol{L})^2 - \frac{1}{2}(tr(\boldsymbol{L}\boldsymbol{L})) \\
&= \frac{1}{2}((tr\boldsymbol{L})^2 - (tr(\boldsymbol{L}\boldsymbol{L}))) \\
&= \frac{1}{2}(L_{ii}L_{jj} - L_{ij}L_{ji}) \ .
\end{aligned}$$

$$I_3(\boldsymbol{L}) = det \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} = \varepsilon_{ijk}L_{i1}L_{j2}L_{k3} = \det \boldsymbol{L} \ .$$

Solution 1.12

With definition (**B**1.159), one finds

$$\begin{aligned}
\boldsymbol{\nabla}(A_{jk}x_jx_k) &= A_{jk}\boldsymbol{\nabla}(x_jx_k) \\
&= A_{jk}(x_k\boldsymbol{\nabla}x_j + x_j\boldsymbol{\nabla}x_k) \\
&= A_{jk}(x_k\delta_{ij} + x_j\delta_{ik}) \\
&= A_{ik}x_k + A_{ji}x_j \\
&= (A_{ij} + A_{ji})x_j\boldsymbol{e}_i \ .
\end{aligned}$$

Solution 1.13

The tensor D_{ij} is decomposed into the sum of a symmetric tensor and of an antisymmetric tensor. One has

$$D_{ij} = D_{ij}^S + D_{ij}^A \ ,$$

with the relations

$$D_{ij}^S = D_{ji}^S$$

and

$$D^A_{ij} = -D^A_{ji} \ .$$

Henceforth, one calculates

$$\begin{aligned} D_{ij}x_ix_j &= (D^S_{ij} + D^A_{ij})x_ix_j \\ &= D^S_{ij}x_ix_j + D^A_{ij}x_ix_j \ . \end{aligned}$$

One uses the fact that the scalar product of the antisymmetric tensor D^A_{ij} and of the symmetric tensor x_ix_j vanishes (cfr. example **B**1.7) to obtain

$$D_{ij}x_ix_j = D^S_{ij}x_ix_j \ .$$

Solution 1.14

As $\boldsymbol{Q}$ is orthogonal, relation (**B**1.92) gives

$$\boldsymbol{Qu} \cdot \boldsymbol{Qv} = \boldsymbol{u} \cdot \boldsymbol{v} \ .$$

By (**B**1.69), one has

$$\boldsymbol{Lu} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{L}^T\boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{Lu} \ .$$

Combining these two last relations, one finds

$$\boldsymbol{Qu} \cdot \boldsymbol{Qv} = \boldsymbol{u} \cdot \boldsymbol{Q}^T\boldsymbol{Qv} = \boldsymbol{u} \cdot \boldsymbol{v}$$

and therefore

$$\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I} \ .$$

Then

$$\boldsymbol{Q}^{-T}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{Q}^{-T}\boldsymbol{I}\boldsymbol{Q}^T = \boldsymbol{Q}^{-T}\boldsymbol{Q}^T$$

one concludes

$$\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I} \ .$$

Multiplying left and right this equation by $\boldsymbol{Q}^{-1}$, one writes

$$\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{Q}^{-1} = \boldsymbol{I}\boldsymbol{Q}^{-1}$$

and consequently

$$\boldsymbol{Q}^T = \boldsymbol{Q}^{-1} \ .$$

Solution 1.15

The antisymmetric tensor $\boldsymbol{L}^A$ has the following matrix

$$[L^A] = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

The first invariant is the sum of the diagonal elements (cf. (**B**1.121))

$$I_1 = 0 \ .$$

For I_2 and I_3, one obtains easily

$$I_2 = \omega_1^2 + \omega_2^2 + \omega_3^2$$

and

$$I_3 = 0 \ .$$

The characteristic equation (**B**1.120) gives

$$\lambda^3 + (\omega_1^2 + \omega_2^2 + \omega_3^2)\lambda = 0$$

or

$$(\lambda^2 + (\omega_1^2 + \omega_2^2 + \omega_3^2))\lambda = 0 \ .$$

The eigenvalues are the roots of this equation. One has

$$\lambda_1 = 0$$

and

$$\lambda_{2,3} = \pm i\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} \ .$$

The system (**B**1.111) produces the equations

$$\begin{aligned} 0 - \omega_3 n_2 + \omega_2 n_3 &= 0 \\ \omega_3 n_1 + 0 - \omega_1 n_3 &= 0 \ . \end{aligned}$$

One extracts

$$\begin{aligned} \frac{n_2}{n_3} &= \frac{\omega_2}{\omega_3} \\ \frac{n_1}{n_3} &= \frac{\omega_1}{\omega_3} \ . \end{aligned}$$

Since the vector $\boldsymbol{n}$ is a unit vector

$$n_1^2 + n_2^2 + n_3^2 = 1 \ .$$

one finds

$$n_3^2\left(\left(\frac{\omega_1}{\omega_3}\right)^2+\left(\frac{\omega_2}{\omega_3}\right)^2+1\right)=1\ .$$

Finally, one obtains

$$\begin{aligned} n_1 &= \omega_1 \\ n_2 &= \omega_2 \\ n_3 &= \omega_3\ . \end{aligned}$$

Solution 1.16

By (**B**1.109) and (**B**1.59), one has successively

$$\boldsymbol{L}\boldsymbol{L}\boldsymbol{n}_i=\lambda_i\boldsymbol{L}\boldsymbol{n}_i$$

$$\boldsymbol{L}^2\boldsymbol{n}_i=\lambda_i\lambda_i\boldsymbol{n}_i=\lambda_i^2\boldsymbol{n}_i$$
$$\boldsymbol{L}^3\boldsymbol{n}_i=\lambda_i^2\boldsymbol{L}\boldsymbol{n}_i=\lambda_i^3\boldsymbol{n}_i\ .$$

Each term of the characteristic equation can be rewritten using the previous relations

$$\begin{aligned} \lambda_i^3\boldsymbol{n}_i &= \boldsymbol{L}^3\boldsymbol{n}_i \\ -I_1\lambda_i^2\boldsymbol{n}_i &= -I_1\boldsymbol{L}^2\boldsymbol{n}_i \\ I_2\lambda_i\boldsymbol{n}_i &= I_2\boldsymbol{L}\boldsymbol{n}_i \\ -I_3\boldsymbol{n}_i &= -I_3\boldsymbol{I}\boldsymbol{n}_i\ . \end{aligned}$$

By adding one finds consecutively

$$\lambda_i^3\boldsymbol{n}_i-I_1\lambda_i^2\boldsymbol{n}_i+I_2\lambda_i\boldsymbol{n}_i-I_3\boldsymbol{n}_i=\boldsymbol{L}^3\boldsymbol{n}_i-I_1\boldsymbol{L}^2\boldsymbol{n}_i+I_2\boldsymbol{L}\boldsymbol{n}_i-I_3\boldsymbol{I}\boldsymbol{n}_i$$

$$(\lambda_i^3-I_1\lambda_i^2+I_2\lambda_i-I_3)\boldsymbol{n}_i=(\boldsymbol{L}^3-I_1\boldsymbol{L}^2+I_2\boldsymbol{L}-I_3\boldsymbol{I})\boldsymbol{n}_i\ .$$

Thus one has

$$\boldsymbol{L}^3-I_1\boldsymbol{L}^2+I_2\boldsymbol{L}-I_3\boldsymbol{I}=0\ .$$

Solution 1.17

Multiplying (**B**1.123) by $\boldsymbol{T}^{-1}$, one obtains

$$\boldsymbol{T}^3\boldsymbol{T}^{-1}-I_1\boldsymbol{T}^2\boldsymbol{T}^{-1}+I_2\boldsymbol{T}\boldsymbol{T}^{-1}-I_3\boldsymbol{I}\boldsymbol{T}^{-1}=\boldsymbol{T}^2-I_1\boldsymbol{T}+I_2\boldsymbol{I}-I_3\boldsymbol{T}^{-1}=\boldsymbol{0}$$

or

$$\boldsymbol{T}^2=I_1\boldsymbol{T}-I_2\boldsymbol{I}+I_3\boldsymbol{T}^{-1}\ .$$

By (**B**1.139) and (**B**1.140), one has

$$\boldsymbol{L} = \boldsymbol{f}(\boldsymbol{T}) = \varphi_0 \boldsymbol{I} + \varphi_1 \boldsymbol{T} + \varphi_2 (I_1 \boldsymbol{T} - I_2 \boldsymbol{I} + I_3 \boldsymbol{T}^{-1}) = \\ = (\varphi_0 - \varphi_2 I_2)\boldsymbol{I} + (\varphi_1 + \varphi_2 I_1)\boldsymbol{T} + \varphi_2 I_3 \boldsymbol{T}^{-1}$$

Setting up

$$\begin{aligned} \alpha_0 &= \varphi_0 - \varphi_2 I_2 \\ \alpha_1 &= \varphi_1 + \varphi_2 I_1 \\ \alpha_2 &= \varphi_2 I_3 \ , \end{aligned}$$

one obtains relation (**B**1.245).

Solution 1.18

Let the matrix $[A]$ be of the order 3 such that

$$[A] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} . \tag{1.4}$$

Its determinant is the scalar given by the relation

$$\begin{aligned} \det[A] &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

1) The product $\varepsilon_{ijk}\varepsilon_{lmn}a_{il}a_{jm}a_{kn}$ generates 36 non vanishing terms (among the 81 possible) that can be grouped in six independent components as the one above. Thus the equation

$$\det[A] = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{lmn}a_{il}a_{jm}a_{kn}$$

is the determinant of $[A]$.

2) By definition, the inverse matrix of $[A]$ is

$$[A]^{-1} = \frac{[M]^T}{\det[A]} , \tag{1.5}$$

where $[M]^T$ is the transpose of the cofactor matrix of $[A]$ with $\det[A] \neq 0$. For matrix $[A]$ given by (1.4) the cofactor matrix has the following elements

$$
\begin{aligned}
M_{11} &= (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & M_{12} &= (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\
M_{13} &= (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & M_{21} &= (-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \\
M_{22} &= (-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & M_{23} &= (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\
M_{31} &= (-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & M_{32} &= (-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\
M_{33} &= (-1)^{3+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \; .
\end{aligned}
$$

Thus one can write

$$
[M] =
\begin{bmatrix}
(a_{22}a_{33} - a_{23}a_{32}) & -(a_{21}a_{33} - a_{23}a_{31}) & (a_{21}a_{32} - a_{22}a_{31}) \\
-(a_{12}a_{33} - a_{13}a_{32}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) \\
(a_{12}a_{23} - a_{13}a_{22}) & -(a_{11}a_{23} - a_{13}a_{21}) & (a_{11}a_{22} - a_{12}a_{21})
\end{bmatrix}
$$

These elements are expressed in index notation using the permutation symbol

$$M_{ij} = \frac{1}{2}\varepsilon_{ikl}\varepsilon_{jmn}a_{km}a_{ln} \; .$$

One can verify easily that expression $\varepsilon_{ikl}\varepsilon_{jmn}$ produces only four non zero terms or two terms that appear twice. This makes the factor one half in the last relation necessary. For example

$$
\begin{aligned}
M_{13} &= [\varepsilon_{123}\varepsilon_{312}a_{21}a_{32} + \varepsilon_{123}\varepsilon_{321}a_{22}a_{31} \\
&\quad + \varepsilon_{132}\varepsilon_{312}a_{31}a_{22} + \varepsilon_{132}\varepsilon_{321}a_{32}a_{21}]/2 \\
&= 2(a_{21}a_{32} - a_{22}a_{31})/2 \; .
\end{aligned}
$$

The transpose of the cofactor matrix $[M]$ is called the adjoint of $[A]$. It is expressed as

$$\left([M]^T\right)_{ij} = \frac{1}{2}\varepsilon_{jkl}\varepsilon_{imn}a_{km}a_{ln} \; . \tag{1.6}$$

Using (1.6) in (1.5), one obtains in index form the elements of the inverse matrix

$$\left([A]^{-1}\right)_{ij} = \frac{1}{2\det[A]}\varepsilon_{jkl}\varepsilon_{imn}a_{km}a_{ln} \; .$$

Solution 1.19

A series of algebraic manipulations leads to the result

$$
\begin{aligned}
\nabla^2(fg) &= \frac{\partial^2(fg)}{\partial x_1^2} + \frac{\partial^2(fg)}{\partial x_2^2} + \frac{\partial^2(fg)}{\partial x_3^2} = \\
&= \frac{\partial}{\partial x_1}\left(\frac{\partial(fg)}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial(fg)}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial(fg)}{\partial x_3}\right) = \\
&= \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}g + f\frac{\partial g}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_2}g + f\frac{\partial g}{\partial x_2}\right) \\
&+ \frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial x_3}g + f\frac{\partial g}{\partial x_3}\right) = \\
&= \frac{\partial^2 f}{\partial x_1^2}g + \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + \frac{\partial^2 g}{\partial x_1^2}f + \\
&+ \frac{\partial^2 f}{\partial x_2^2}g + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_2} + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_2} + \frac{\partial^2 g}{\partial x_2^2}f + \\
&+ \frac{\partial^2 f}{\partial x_3^2}g + \frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_3} + \frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_3} + \frac{\partial^2 g}{\partial x_3^2}f = \\
&= \left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}\right)g + \left(\frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 g}{\partial x_2^2} + \frac{\partial^2 g}{\partial x_3^2}\right)f \\
&+ 2\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + 2\frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_2} + 2\frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_3} = \\
&= f\nabla^2 g + g\nabla^2 f + 2\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g \ .
\end{aligned}
$$

Chapitre 2

Kinematics of continuous media

Solution 2.1

With relation (**B**2.8), one writes

$$\boldsymbol{U}(\boldsymbol{X},t) = \boldsymbol{x} - \boldsymbol{X} = -\frac{1}{2} X_1 \boldsymbol{e}_1$$

and by (**B**2.9), one has

$$\boldsymbol{u}(\boldsymbol{x},t) = -x_1 \boldsymbol{e}_1 \ .$$

Solution 2.2

By (**B**2.133), one has

$$x_1 = X_1 + kX_2 \qquad x_2 = X_2 \qquad x_3 = X_3 \ .$$

Recalling the definition of the deformation gradient tensor (**B**2.67)

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \ ,$$

one calculates successively

$$\boldsymbol{F} = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the right Cauchy-Green deformation tensor

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k & 0 \\ k & k^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the left Cauchy-Green deformation tensor

$$\boldsymbol{c} = \boldsymbol{F}\boldsymbol{F}^T = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+k^2 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the Green-Lagrange deformation tensor

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2} \begin{pmatrix} 0 & k & 0 \\ k & k^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the Euler-Almansi deformation tensor

$$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{I} - \boldsymbol{c}^{-1})$$

$$\boldsymbol{c}^{-1} = \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}$$

$$\boldsymbol{F}^{-1} = \begin{pmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{F}^{-T} = \begin{pmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{c}^{-1} = \boldsymbol{F}^{-T}\boldsymbol{F}^{-1} = \begin{pmatrix} 1 & -k & 0 \\ -k & k^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{e} = \frac{1}{2} \begin{pmatrix} 0 & k & 0 \\ k & -k^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

We detail below the computation of $\boldsymbol{F}^{-1}$ by the adjoint and cofactors method

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & k \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{12} = (-1)^3 \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$A_{13} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{21} = (-1)^3 \begin{vmatrix} k & 0 \\ 0 & 1 \end{vmatrix} = -k$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & k \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{31} = (-1)^4 \begin{vmatrix} k & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A}^T = \begin{pmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|\boldsymbol{M}| = \begin{vmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\boldsymbol{F}^{-1} = \frac{\boldsymbol{A}^T}{|\boldsymbol{M}|} = \begin{pmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{F} \cdot \boldsymbol{F}^{-1} = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -k+k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}$$

Solution 2.3

$$\boldsymbol{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}$$

$$\boldsymbol{M}^{-1} = \begin{pmatrix} m^{-1} & 0 & 0 \\ 0 & m^{-1} & 0 \\ 0 & 0 & m^{-1} \end{pmatrix}$$

As

$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

one obtains with (**B**2.130)

$$\boldsymbol{F} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}$$

and the right Cauchy-Green deformation tensor as

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} = \begin{pmatrix} m^2 & 0 & 0 \\ 0 & m^2 & 0 \\ 0 & 0 & m^2 \end{pmatrix} .$$

Let us remark that $\boldsymbol{C} = \boldsymbol{c}$ as tensors $\boldsymbol{F}$ and $\boldsymbol{F}^T$ are diagonal. The following tensors can be easily expressed :

the Green-Lagrange deformation tensor

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2} \begin{pmatrix} m^2 - 1 & 0 & 0 \\ 0 & m^2 - 1 & 0 \\ 0 & 0 & m^2 - 1 \end{pmatrix}$$

the inverse of the left Cauchy-Green tensor

$$\boldsymbol{c}^{-1} = \boldsymbol{F}^{-T} \boldsymbol{F}^{-1} = \begin{pmatrix} m^{-2} & 0 & 0 \\ 0 & m^{-2} & 0 \\ 0 & 0 & m^{-2} \end{pmatrix}$$

the Euler-Almansi deformation tensor

$$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{I} - \boldsymbol{C}^{-1}) = \frac{1}{2} \begin{pmatrix} 1 - m^{-2} & 0 & 0 \\ 0 & 1 - m^{-2} & 0 \\ 0 & 0 & 1 - m^{-2} \end{pmatrix} .$$

Solution 2.4

1) The trajectory is the spatial curve describing the successive positions $\boldsymbol{x}$ of a particle $\boldsymbol{X}$ with respect to time t. One eliminates the variable t in the relation $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ in such a way as to obtain a system of implicit equations linking the positions x_i. Thus

$$\begin{aligned} \left(\frac{x_1 - X_1}{a}\right)^2 + \left(\frac{x_2 - X_2}{b}\right)^2 &= \cos^2 2\pi(\frac{t}{T} - \frac{X_1}{L}) \\ &+ \sin^2 2\pi(\frac{t}{T} - \frac{X_1}{L}) = 1 \\ x_3 &= X_3 \ . \end{aligned}$$

The trajectory of a particle with given material coordinates X_1, X_2, X_3 is located on the plane $x_3 = X_3$. In this plane, the trajectory is an ellipse whose centre is at point with coordinates X_1, X_2, X_3 and whose principal axes are oriented in the directions $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ and have $2a$ and $2b$ as respective lengths.

2) As the motion is given in the Lagrangian description, one finds the velocity and acceleration components by taking the partial derivative with respect to time with the X_j held fixed. One finds

$$\begin{aligned}
V_1 &= \frac{\partial x_1}{\partial t}\,|_{X_j} = -\frac{2\pi a}{T}\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})\\
V_2 &= \frac{\partial x_2}{\partial t}\,|_{X_j} = \frac{2\pi b}{T}\cos 2\pi(\frac{t}{T}-\frac{X_1}{L})\\
V_3 &= \frac{\partial x_3}{\partial t}\,|_{X_j} = 0
\end{aligned}$$

$$\begin{aligned}
A_1 &= \frac{\partial V_1}{\partial t}\,|_{X_j} = -\frac{4\pi^2 a}{T^2}\cos 2\pi(\frac{t}{T}-\frac{X_1}{L})\\
A_2 &= \frac{\partial V_2}{\partial t}\,|_{X_j} = -\frac{4\pi^2 b}{T^2}\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})\\
A_3 &= \frac{\partial V_3}{\partial t}\,|_{X_j} = 0\ .
\end{aligned}$$

3) The deformation gradient tensor is obtained by the relation $F_{ij} = \frac{\partial x_i}{\partial X_j}$. Thus one has

$$[F] = \begin{pmatrix} 1+\frac{2\pi a}{L}\sin 2\pi(\frac{t}{T}-\frac{X_1}{L}) & 0 & 0\\ -\frac{2\pi b}{L}\cos 2\pi(\frac{t}{T}-\frac{X_1}{L}) & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\ .$$

In Lagrangian representation, one writes $\frac{D\boldsymbol{F}}{Dt} = \frac{\partial}{\partial t}F_{ij}\,|_{X_k}$. The matrix $[\dot{F}]$ gives

$$[\dot{F}] = \begin{pmatrix} \frac{4\pi^2 a}{LT}\cos 2\pi(\frac{t}{T}-\frac{X_1}{L}) & 0 & 0\\ \frac{4\pi^2 b}{LT}\sin 2\pi(\frac{t}{T}-\frac{X_1}{L}) & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\ .$$

4) Using relation (**B**2.179), $\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}$, one has $\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$. As on the one hand, $det(\boldsymbol{F}) = 1+\frac{2\pi a}{L}\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})$ and on the other hand, the adjoint of $[F]$ is written by setting $2\pi(\frac{t}{T}-\frac{X_1}{L}) = arg$

$$adj[F] = \begin{pmatrix} 1 & 0 & 0\\ \frac{2\pi b}{L}\cos arg & 1+\frac{2\pi a}{L}\sin arg & 0\\ 0 & 0 & 1+\frac{2\pi a}{L}\sin arg\end{pmatrix}\ ,$$

one obtains

$$[F^{-1}] = \frac{1}{det(\boldsymbol{F})} adj[F] = \begin{pmatrix} \frac{1}{1+\frac{2\pi a}{L}\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})} & 0 & 0 \\ \frac{\frac{2\pi b}{L}\cos 2\pi(\frac{t}{T}-\frac{X_1}{L})}{1+\frac{2\pi a}{L}\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

One thus finds

$$[L] = \begin{pmatrix} \frac{4\pi^2 a\cos 2\pi(\frac{t}{T}-\frac{X_1}{L})}{LT+2\pi aT\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})} & 0 & 0 \\ \frac{4\pi^2 b\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})}{LT+2\pi aT\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

5) the tensors $\boldsymbol{d}$ et $\dot{\boldsymbol{\omega}}$ are respectively the symmetric and antisymmetric parts of $\boldsymbol{L}$, cf. (**B**2.184). One finds

$$[d] = \begin{pmatrix} \frac{4\pi^2 a\cos 2\pi(\frac{t}{T}-\frac{X_1}{L})}{LT+2\pi aT\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})} & \frac{2\pi^2 b\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})}{LT+2\pi aT\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})} & 0 \\ \frac{2\pi^2 b\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})}{LT+2\pi aT\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$[\dot{\boldsymbol{\omega}}] = \begin{pmatrix} 0 & \frac{-2\pi^2 b\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})}{LT+2\pi aT\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})} & 0 \\ \frac{2\pi^2 b\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})}{LT+2\pi aT\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

The evaluation of the components of the vorticity vector is based on relation (**B**2.187). One calculates

$$\begin{aligned} \dot{\Omega}_1 &= 0 \\ \dot{\Omega}_2 &= 0 \\ \dot{\Omega}_3 = \dot{\omega}_{21} &= \frac{2\pi^2 b\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})}{LT+2\pi aT\sin 2\pi(\frac{t}{T}-\frac{X_1}{L})} . \end{aligned}$$

Solution 2.5

1) With the definition of the deformation gradient tensor (**B**2.67), we write

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{pmatrix} 1 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

As tensor $\boldsymbol{F}$ is independent of $\boldsymbol{X}$, the deformation is homogeneous. As $\det \boldsymbol{F} \neq 1$, this deformation is not isochoric, as we will notice in chapter 3, (cf. (**B**3.38)). For the transformation to be invertible, it is necessary to verify the inequalities (**B**2.69). This implies $-1 < a < +1$.

2) One calculates successively :
the right Cauchy-Green deformation tensor

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \begin{pmatrix} 1+a^2 & 2a & 0 \\ 2a & 1+a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the Green-Lagrange deformation tensor

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \begin{pmatrix} \frac{a^2}{2} & a & 0 \\ a & \frac{a^2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the gradient of the displacement vector

$$\boldsymbol{\nabla u} = \boldsymbol{F} - \boldsymbol{I} = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the infinitesimal deformation tensor

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T) = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

With the hypothesis

$$a << 1$$

the tensors $\boldsymbol{C}, \boldsymbol{E}, \boldsymbol{\varepsilon}$ become

$$\boldsymbol{C} = \begin{pmatrix} 1+a^2 & 2a & 0 \\ 2a & 1+a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2a & 0 \\ 2a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boldsymbol{\varepsilon} .$$

3) The unit eigenvectors of $\boldsymbol{C}$ are given by the relation (**B**2.110). Let's express the vectors oriented in direction x_3 and the diagonals AH and DE. One has

$$\boldsymbol{A}_1 = \alpha(\boldsymbol{x}_1 + \boldsymbol{x}_2)$$
$$\boldsymbol{A}_2 = \beta(\boldsymbol{x}_1 - \boldsymbol{x}_2)$$
$$\boldsymbol{A}_3 = \gamma\boldsymbol{x}_3 .$$

By (**B**2.120), one has

$$\boldsymbol{C}\boldsymbol{A}_i = \lambda_i^2 \boldsymbol{A}_i \qquad \text{(no summation on } i\text{)}$$

and therefore, one evaluates successively

$$\begin{aligned}
\boldsymbol{C}\boldsymbol{A}_1 &= \begin{pmatrix} 1+a^2 & 2a & 0 \\ 2a & 1+a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha \\ 0 \end{pmatrix} = (1+a)^2 \begin{pmatrix} \alpha \\ \alpha \\ 0 \end{pmatrix} \\
&= (1+a)^2 \boldsymbol{A}_1 \\
\boldsymbol{C}\boldsymbol{A}_2 &= \begin{pmatrix} 1+a^2 & 2a & 0 \\ 2a & 1+a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ -\beta \\ 0 \end{pmatrix} = (1-a)^2 \begin{pmatrix} \beta \\ -\beta \\ 0 \end{pmatrix} \\
&= (1-a)^2 \boldsymbol{A}_2 \\
\boldsymbol{C}\boldsymbol{A}_3 &= \begin{pmatrix} 1+a^2 & 2a & 0 \\ 2a & 1+a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} = \boldsymbol{A}_3 \ .
\end{aligned}$$

The eignevaleurs of $\boldsymbol{C}$ are

$$\begin{aligned}
\lambda_1^2 &= (1+a)^2 \\
\lambda_2^2 &= (1-a)^2 \\
\lambda_3^2 &= 1
\end{aligned}$$

By the spectral representation $\boldsymbol{C}$, one writes

$$\boldsymbol{C} = \lambda_i^2 (\boldsymbol{A}_i \otimes \boldsymbol{A}_i) = \lambda_1^2 (\boldsymbol{A}_1 \otimes \boldsymbol{A}_1) + \lambda_2^2 (\boldsymbol{A}_2 \otimes \boldsymbol{A}_2) + \lambda_3^2 (\boldsymbol{A}_3 \otimes \boldsymbol{A}_3) \ .$$

One has

$$\boldsymbol{A}_1 \otimes \boldsymbol{A}_1 = \begin{pmatrix} \alpha \\ \alpha \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha^2 & \alpha^2 & 0 \\ \alpha^2 & \alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{A}_2 \otimes \boldsymbol{A}_2 = \begin{pmatrix} \beta \\ -\beta \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \beta \\ -\beta \\ 0 \end{pmatrix} = \begin{pmatrix} \beta^2 & -\beta^2 & 0 \\ -\beta^2 & \beta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{A}_3 \otimes \boldsymbol{A}_3 = \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ .$$

One verifies

$$[C] = (1+a)^2 \times \begin{pmatrix} \alpha^2 & \alpha^2 & 0 \\ \alpha^2 & \alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (1-a)^2 \times \begin{pmatrix} \beta^2 & -\beta^2 & 0 \\ -\beta^2 & \beta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+1 \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix} =$$

$$= (1+a)^2 \times \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + (1-a)^2 \times \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+1 \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+a^2 & 2a & 0 \\ 2a & 1+a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

By (**B**.2.109), one obtains easily

$$[U] = \begin{pmatrix} 1 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [F] .$$

4)
$$\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1} = \boldsymbol{F}\boldsymbol{F}^{-1} = \boldsymbol{I} .$$

Solution 2.6

For the right Cauchy-Green tensor, one carries out the algebra

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F};\ \boldsymbol{C}^* = \boldsymbol{F}^{*T}\boldsymbol{F}^*;\ \boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F};\ \boldsymbol{F}^{*T} = \boldsymbol{F}^T\boldsymbol{Q}^T = \boldsymbol{F}^T\boldsymbol{Q}^{-1}$$

$$\boldsymbol{C}^* = \boldsymbol{F}^T\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{F} = \boldsymbol{F}^T\boldsymbol{I}\boldsymbol{F} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{C} .$$

For the Green-Lagrange tensor, one finds

$$2\boldsymbol{E} = \boldsymbol{C} - \boldsymbol{I};\ \boldsymbol{C}^* = \boldsymbol{C}$$

$$2\boldsymbol{E}^* = 2\boldsymbol{E} \Rightarrow \boldsymbol{E}^* = \boldsymbol{E} .$$

For the left Cauchy-Green tensor, one obtains

$$\boldsymbol{c} = \boldsymbol{F}\boldsymbol{F}^T;\ \boldsymbol{c}^* = \boldsymbol{F}^*\boldsymbol{F}^{*T};\ \boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F};\ \boldsymbol{F}^{*T} = \boldsymbol{F}^T\boldsymbol{Q}^T = \boldsymbol{F}^T\boldsymbol{Q}^{-1}$$

$$\boldsymbol{c}^* = \boldsymbol{F}^*\boldsymbol{F}^{*T} = \boldsymbol{Q}\boldsymbol{F}\boldsymbol{F}^T\boldsymbol{Q}^T = \boldsymbol{Q}\boldsymbol{c}\boldsymbol{Q}^T .$$

As a consequence, $\boldsymbol{c}$ is spatially objective. For the Euler-Almansi tensor, one verifies

$$2\boldsymbol{e}^* = \boldsymbol{I} - \boldsymbol{c}^{*-1};\ \boldsymbol{Q}^{-T} = \boldsymbol{Q};\ \boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$$

$$2\boldsymbol{e}^* = \boldsymbol{I} - \boldsymbol{c}^{*-1} = \boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T - \boldsymbol{Q}^{-T}\boldsymbol{c}^{-1}\boldsymbol{Q}^{-1} = 2\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T .$$

Solution 2.7

With (**B**2.205) and the orthogonality of $\boldsymbol{R}$, one has

$$\boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F} \quad \text{and} \quad \boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I} \ .$$

By the polar decomposition theorem (**B**1.132), one writes

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R};\ \boldsymbol{F}^* = \boldsymbol{R}^*\boldsymbol{U}^* = \boldsymbol{V}^*\boldsymbol{R}^* = \boldsymbol{Q}\boldsymbol{F}$$

$$\boldsymbol{R}^*\boldsymbol{U}^* = \boldsymbol{Q}\boldsymbol{R}\boldsymbol{U}$$

$$\boldsymbol{R}^* = \boldsymbol{Q}\boldsymbol{R}\boldsymbol{U}\boldsymbol{U}^{*-1} \ .$$

This relation is trivially satisfied if we set $\boldsymbol{R}^* = \boldsymbol{Q}\boldsymbol{R}$ and thus $\boldsymbol{U}^* = \boldsymbol{U}$. Similarly one has successively

$$\boldsymbol{V}^*\boldsymbol{R}^* = \boldsymbol{Q}\boldsymbol{V}\boldsymbol{R}$$

and thus,

$$\boldsymbol{V}^* = \boldsymbol{Q}\boldsymbol{V}\boldsymbol{R}\boldsymbol{R}^{*-1} = \boldsymbol{Q}\boldsymbol{V}(\boldsymbol{R}\boldsymbol{R}^T\boldsymbol{Q}^T) = \boldsymbol{Q}\boldsymbol{V}\boldsymbol{Q}^T \ .$$

Solution 2.8

With the help of (**B**2.88), (**B**2.91), (**B**2.179) and (**B**2.180), one has

$$2\boldsymbol{E} = \boldsymbol{C} - \boldsymbol{I};\ \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F};\ \dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F};\ 2\boldsymbol{d} = \boldsymbol{L} + \boldsymbol{L}^T \ .$$

Therefore, one finds successively

$$\dot{\boldsymbol{E}} = \frac{\dot{\boldsymbol{C}}}{2}$$

$$\dot{\boldsymbol{C}} = \boldsymbol{F}^T\dot{\boldsymbol{F}} + \dot{\boldsymbol{F}}^T\boldsymbol{F} = \boldsymbol{F}^T\boldsymbol{L}\boldsymbol{F} + \boldsymbol{F}^T\boldsymbol{L}^T\boldsymbol{F} = \boldsymbol{F}^T(\boldsymbol{L} + \boldsymbol{L}^T)\boldsymbol{F} = 2\boldsymbol{F}^T\boldsymbol{d}\boldsymbol{F}$$

$$\dot{\boldsymbol{E}} = \boldsymbol{F}^T\boldsymbol{d}\boldsymbol{F} \ .$$

Solution 2.9

The motion described by the relations

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3 \ ,$$

leads to the deformation gradient tensor defined by (**B**2.67)

$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

as the diagonal tensor

$$F_{ij} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} .$$

The matrix of the right Cauchy-Green deformation tensor is

$$\begin{aligned}[C] = [F]^T[F] &= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}\end{aligned}$$

and the matrix of the right Green-Lagrange deformation tensor given by

$$[E] = \frac{1}{2}([C] - [I]) = \frac{1}{2} \begin{pmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_3^2 - 1 \end{pmatrix} .$$

By (**B**2.88)

$$\boldsymbol{C} = \boldsymbol{U}^2$$

one finds

$$[U] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

and thus,

$$\begin{cases} [F] = [U] \\ [F] = [R][U] \end{cases}$$

and

$$[R] = [I] .$$

Solution 2.10

With (**B**2.91) and (**B**2.92), one has

$$2\boldsymbol{E} = \boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I} \quad \text{and} \quad 2\boldsymbol{e} = \boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1} .$$

The last relation is multiplied left by $\boldsymbol{F}^{-T}\boldsymbol{F}^T(= \boldsymbol{I})$ and right by $\boldsymbol{F}\boldsymbol{F}^{-1}$ to obtain successively

$$\begin{aligned}2\boldsymbol{e} &= \boldsymbol{F}^{-T}(\boldsymbol{F}^T(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1})\boldsymbol{F})\boldsymbol{F}^{-1} \\ &= \boldsymbol{F}^{-T}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{F}^T\boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\boldsymbol{F})\boldsymbol{F}^{-1} \\ &= \boldsymbol{F}^{-T}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})\boldsymbol{F}^{-1} \\ &= 2\boldsymbol{F}^{-T}\boldsymbol{E}\boldsymbol{F}^{-1} .\end{aligned}$$

By the polar decomposition theorem, one has

$$\boldsymbol{F}^T = \boldsymbol{U}^T \boldsymbol{R}^T = \boldsymbol{U}\boldsymbol{R}^T \ .$$

By definition (**B**2.89), one gets

$$\boldsymbol{c} = \boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{R}\boldsymbol{U}\boldsymbol{U}\boldsymbol{R}^T = \boldsymbol{R}\boldsymbol{U}^2\boldsymbol{R}^T = \boldsymbol{R}\boldsymbol{C}\boldsymbol{R}^T \ .$$

Solution 2.11

We recall relation (**B**2.108)

$$\boldsymbol{U}\boldsymbol{A}_i = \lambda_i \boldsymbol{A}_i \ .$$

With (**B**2.112) one finds

$$\boldsymbol{R}\boldsymbol{U}\boldsymbol{A}_i = \lambda_i \boldsymbol{R}\boldsymbol{A}_i; \ \ \boldsymbol{F}\boldsymbol{A}_i = \lambda_i \boldsymbol{b}_i \ .$$

With (**B**2.109)

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \boldsymbol{A}_i \otimes \boldsymbol{A}_i \ ,$$

we find

$$\boldsymbol{R}\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i (\boldsymbol{R}\boldsymbol{A}_i) \otimes \boldsymbol{A}_i \quad \text{et} \quad \boldsymbol{F} = \sum_{i=1}^{3} \lambda_i \boldsymbol{b}_i \otimes \boldsymbol{A}_i \ .$$

Solution 2.12

Equation (**B**2.106) can be written as $\boldsymbol{F}^T \boldsymbol{n} ds = J\boldsymbol{N} dS$ or $F_{ji} n_j ds = J N_i dS$.

Introduce (**B**2.147) in this relation to obtain

$$(\delta_{ji} + O(\varepsilon))\, n_j ds = (1 + O(\varepsilon))\, N_i dS \ , \tag{2.1}$$

which yields

$$n_i ds \approx N_i dS \ . \tag{2.2}$$

2.13

Lengthy solution.

(a) Use (**B**2.77), (**B**2.70), (**B**2.88) and (**B**2.120) to obtain

$$
\begin{aligned}
F_{mi}F_{mj} &= \left(\delta_{mi} + \frac{\partial U_m}{\partial X_i}\right)\left(\delta_{mj} + \frac{\partial U_m}{\partial X_j}\right) \\
&= \delta_{mi}\delta_{mj} + \delta_{mi}\frac{\partial U_m}{\partial X_j} + \delta_{mj}\frac{\partial U_m}{\partial X_i} + \delta_{mi}\delta_{mj}\frac{\partial U_m}{\partial X_j}\frac{\partial U_m}{\partial X_i} \\
&= \delta_{ij} + \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) + \frac{\partial U_i}{\partial X_j}\frac{\partial U_j}{\partial X_i} \\
&= \delta_{ij} + \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) + O(\varepsilon^2)
\end{aligned}
$$

Then one evaluates

$$
\begin{aligned}
U_{ij} &= (F_{mi}F_{mj})^{1/2} = \left(\delta_{ij} + \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) + \frac{\partial U_i}{\partial X_j}\frac{\partial U_j}{\partial X_i}\right)^{1/2} \\
&\approx \delta_{ij} + \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) + O(\varepsilon^2) \\
&\approx \delta_{ij} + \varepsilon_{ij} + O(\varepsilon^2)\ .
\end{aligned}
$$

In the last relation we used $(1+a)^n \approx 1 + na$ for $a \ll 1$. Therefore

$$\boldsymbol{U} \approx \boldsymbol{I} + \boldsymbol{\varepsilon}\ .$$

(b) It is easy to show that

$$U_{kj}^{-1} = \delta_{kj} - \frac{1}{2}\left(\frac{\partial U_k}{\partial X_j} + \frac{\partial U_j}{\partial X_k}\right) + O(\varepsilon^2)$$

is the inverse of U_{ij}, i.e. $U_{ik}U_{kj}^{-1} \approx \delta_{ij}$.

Then use (**B**2.73) as follows

$$
\begin{aligned}
R_{ij} &= F_{ik}U_{kj}^{-1} = \left(\delta_{ik} + \frac{\partial U_i}{\partial X_k}\right)\left(\delta_{kj} - \frac{1}{2}\left(\frac{\partial U_k}{\partial X_j} + \frac{\partial U_j}{\partial X_k}\right) + O(\varepsilon^2)\right) \\
&= \delta_{ij} - \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) + O(\varepsilon^2) + \frac{\partial U_i}{\partial X_j} \\
&\quad - \frac{1}{2}\frac{\partial U_i}{\partial X_k}\left(\frac{\partial U_k}{\partial X_j} + \frac{\partial U_j}{\partial X_k}\right) + O(\varepsilon^2) \\
&= \delta_{ij} - \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} - 2\frac{\partial U_i}{\partial X_j}\right) + O(\varepsilon^2) \\
&\quad - \frac{1}{2}\frac{\partial U_i}{\partial X_k}\left(\frac{\partial U_k}{\partial X_j} + \frac{\partial U_j}{\partial X_k}\right) + O(\varepsilon^2) \\
&= \delta_{ij} + \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} - \frac{\partial U_j}{\partial X_i}\right) - \frac{1}{2}\frac{\partial U_i}{\partial X_k}\left(\frac{\partial U_k}{\partial X_j} + \frac{\partial U_j}{\partial X_k}\right) + O(\varepsilon^2) \\
&= \delta_{ij} + \omega_{ij} + O(\varepsilon^2)\ .
\end{aligned}
$$

Therefore

$$\boldsymbol{R} \approx \boldsymbol{I} + \boldsymbol{\omega} \ .$$

Short version

(a) Use (**B**2.77), (**B**2.70), (**B**2.88) and (**B**2.120) to obtain

$$\begin{aligned} F_{mi}F_{mj} &= \left(\delta_{mi} + \frac{\partial U_m}{\partial X_i}\right)\left(\delta_{mj} + \frac{\partial U_m}{\partial X_j}\right) \\ &= \delta_{mi}\delta_{mj} + \delta_{mi}\frac{\partial U_m}{\partial X_j} + \delta_{mj}\frac{\partial U_m}{\partial X_i} + \delta_{mi}\delta_{mj}\frac{\partial U_m}{\partial X_j}\frac{\partial U_m}{\partial X_i} \\ &\approx \delta_{ij} + \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) \end{aligned}$$

Then we evaluate

$$\begin{aligned} U_{ij} &= (F_{mi}F_{mj})^{1/2} = \left(\delta_{ij} + \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right)\right)^{1/2} \\ &\approx \delta_{ij} + \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) \\ &\approx \delta_{ij} + \varepsilon_{ij} \end{aligned}$$

In the last relation we used $(1+a)^n \approx 1 + na$ for $a \ll 1$. Therefore

$$\boldsymbol{U} \approx \boldsymbol{I} + \boldsymbol{\varepsilon} \ .$$

(b) It is easy to show that

$$U^{-1}_{kj} = \delta_{kj} - \frac{1}{2}\left(\frac{\partial U_k}{\partial X_j} + \frac{\partial U_j}{\partial X_k}\right) + O(\varepsilon^2)$$

is the inverse of U_{ij}, i.e. $U_{ik}U^{-1}_{kj} \approx \delta_{ij}$.

Then use (**B**2.73) as follows

$$\begin{aligned} R_{ij} &= F_{ik}U^{-1}_{kj} = \left(\delta_{ik} + \frac{\partial U_i}{\partial X_k}\right)\left(\delta_{kj} - \frac{1}{2}\left(\frac{\partial U_k}{\partial X_j} + \frac{\partial U_j}{\partial X_k}\right)\right) \\ &\approx \delta_{ij} - \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) + \frac{\partial U_i}{\partial X_j} \\ &\approx \delta_{ij} - \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} - 2\frac{\partial U_i}{\partial X_j}\right) \\ &= \delta_{ij} + \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} - \frac{\partial U_j}{\partial X_i}\right) \\ &= \delta_{ij} + \omega_{ij} \end{aligned}$$

to obtain

$$\boldsymbol{R} \approx \boldsymbol{I} + \boldsymbol{\omega} \ .$$

Solution 2.14

Relation (**B**2.157) is given by $\cos\gamma_{12} \cong 2\varepsilon_{12}$, where γ_{12} is the angle between two vectors initially (before deformation) orthogonal. The variables ε_{ij} are the components of the infinitesimal deformation tensor. The components of vectors before and after the movement (or deformation) are given by the following relations with reference to figure 2.1

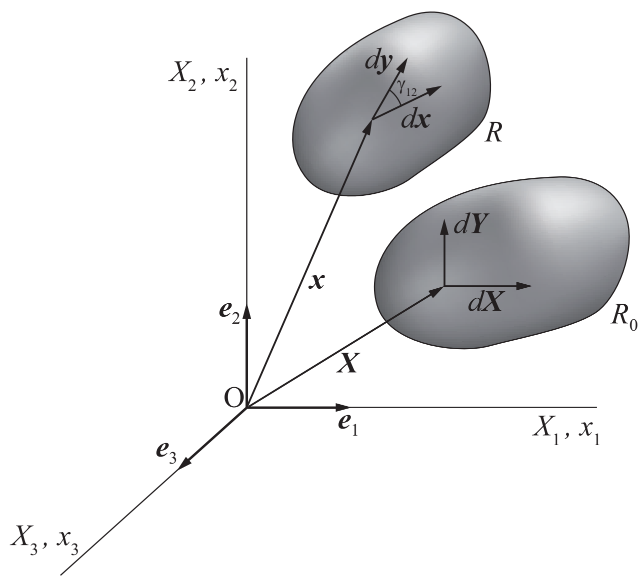

Figure 2.1 Modification of the angles between two vectors.

$$d\boldsymbol{X} : (dX_1, 0, 0) \rightarrow d\boldsymbol{x} : (dx_1, dx_2, dx_3) \tag{2.3}$$
$$d\boldsymbol{Y} : (0, dY_2, 0) \rightarrow d\boldsymbol{y} : (dy_1, dy_2, dy_3) \ . \tag{2.4}$$

According to the body motion, one has from (**B**2.8)

$$dx_i = dU_i + dX_i \ . \tag{2.5}$$

We remplace dU_i by the next expression (see top of p. 90 of the book)

$$dU_i = \varepsilon_{ij} dX_j + \omega_{ij} dX_j \ .$$

To simplify the algebra, we assume that the infinitesimal rotations vanish. Thus

$$\omega_{ij} = 0 \Rightarrow dU_i = \varepsilon_{ij} dX_j \ .$$

The relation (2.5) becomes, taking into account (2.3)

$$dx_i = dX_i + \varepsilon_{ij} dX_j = (\delta_{ij} + \varepsilon_{ij})\, dX_j = (\delta_{i1} + \varepsilon_{i1})\, dX_1$$

and then

$$dx_1 = (1 + \varepsilon_{11}) dX_1, \quad dx_2 = \varepsilon_{21} dX_1, \quad dx_3 = \varepsilon_{31} dX_1 \ .$$

Similarly, we obtain for segment $d\boldsymbol{y}$

$$dy_i = dY_i + \varepsilon_{ij} dY_j = (\delta_{ij} + \varepsilon_{ij})\, dY_j = (\delta_{i2} + \varepsilon_{i2})\, dY_2$$

and then

$$dy_1 = \varepsilon_{12} dY_2, \quad dy_2 = (1 + \varepsilon_{22}) dY_2, \quad dy_3 = \varepsilon_{32} dY_2 \ .$$

By (**B**2.157), one writes

$$\begin{aligned}
\cos\gamma_{12} &= \frac{d\boldsymbol{x} \cdot d\boldsymbol{y}}{\|d\boldsymbol{x}\|\,\|d\boldsymbol{y}\|} \\
&= \frac{(1+\varepsilon_{11})\varepsilon_{12} + (1+\varepsilon_{22})\varepsilon_{21} + \varepsilon_{31}\varepsilon_{32}}{\left[(1+\varepsilon_{11})^2 + \varepsilon_{21}^2 + \varepsilon_{31}^2\right]^{1/2} \left[\varepsilon_{12}^2 + (1+\varepsilon_{22})^2 + \varepsilon_{32}^2\right]^{1/2}} \\
&= \frac{\varepsilon_{12} + \varepsilon_{11}\varepsilon_{12} + \varepsilon_{21} + \varepsilon_{21}\varepsilon_{22} + \varepsilon_{31}\varepsilon_{32}}{\left[1 + \varepsilon_{11}^2 + 2\varepsilon_{11} + \varepsilon_{21}^2 + \varepsilon_{31}^2\right]^{1/2} \left[\varepsilon_{12}^2 + 1 + \varepsilon_{22}^2 + 2\varepsilon_{22} + \varepsilon_{32}^2\right]^{1/2}} \ .
\end{aligned}$$

Neglecting the terms of order greater than 1 (products of components ε_{ij}) and taking into account the symmetry $\varepsilon_{ij} = \varepsilon_{ji}$, one obtains (**B**2.157)

$$\begin{aligned}
\cos\gamma_{12} &= \frac{d\boldsymbol{x} \cdot d\boldsymbol{y}}{\|d\boldsymbol{x}\|\,\|d\boldsymbol{y}\|} \approx \frac{2\varepsilon_{12}}{[1+2\varepsilon_{11}]^{1/2}\,[1+2\varepsilon_{22}]^{1/2}} \approx \frac{2\varepsilon_{12}}{(1+\varepsilon_{11})\,(1+\varepsilon_{22})} \\
&\approx 2\varepsilon_{12}\,(1-\varepsilon_{11})\,(1-\varepsilon_{22}) = 2\varepsilon_{12}\,(1-\varepsilon_{11}-\varepsilon_{22}-\varepsilon_{11}\varepsilon_{22}) \approx 2\varepsilon_{12} \ .
\end{aligned}$$

Thus we have

$$\cos\gamma_{12} = 2\varepsilon_{12}$$

and consequently

$$\cos\gamma_{12} = \sin\varphi_{12} \approx \varphi_{12} = 2\varepsilon_{12} \ ,$$

as $\gamma_{12} + \varphi_{12} = \pi/2$.

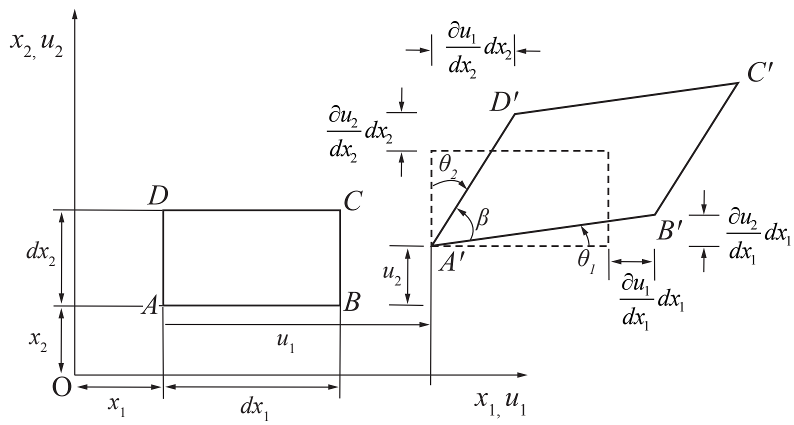

Figure 2.2 Deformation of an infinitesimal element.

Solution 2.15

Let $ABCD$ be an infinitesimal element with sides $dx_1\,dx_2$ given in figure 2.2 (**B**2.23 in the book).

As angle θ_1 is small, we have the approximation

$$\tan\theta_1 \approx \theta_1$$

Inspecting figure 2.2, we find

$$\theta_1 = \frac{\frac{\partial u_2}{\partial x_1} dx_1}{dx_1 + \frac{\partial u_1}{\partial x_1} dx_1}$$

and thus

$$\theta_1 = \frac{\partial u_2}{\partial x_1} \ .$$

By a similar line of reasoning, one has

$$\theta_2 = \frac{\partial u_1}{\partial x_2} \ .$$

We know that

$$\theta_1 + \theta_2 = \phi_{12} = \frac{\pi}{2} - \gamma_{12} \ .$$

Consequently, one finds

$$\cos\gamma_{12} = \sin\phi_{12} = \sin(\theta_1+\theta_2) \approx \theta_1+\theta_2 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = 2\varepsilon_{12} \ .$$

By the definition of the relative extension (**B**2.153), we write

$$\varepsilon_{11} = \frac{A'B' - AB}{AB} = \frac{A'B' - dx_1}{dx_1}$$

and

$$\varepsilon_{22} = \frac{A'D' - AD}{AD} = \frac{A'D' - dx_2}{dx_2} \ .$$

Next we obtain the relations

$$(A'B')^2 = (dx_1(1+\varepsilon_{11}))^2 = \left(dx_1 + \frac{\partial u_1}{\partial x_1}dx_1\right)^2 + \left(\frac{\partial u_2}{\partial x_1}dx_1\right)^2$$

$$dx_1^2(\varepsilon_{11}^2 + 2\varepsilon_{11} + 1) = dx_1^2\left(1 + 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2\right)$$

$$2\varepsilon_{11} \approx 2\frac{\partial u_1}{\partial x_1} \ .$$

Thus

$$\varepsilon_{11} \approx \frac{\partial u_1}{\partial x_1}$$

and for $(A'D')$ one has

$$\varepsilon_{22} \approx \frac{\partial u_2}{\partial x_2} \ .$$

Solution 2.16

By relation (**B**2.80), within the framework of small deformations, one has

$$ds^2 - dS^2 = 2E_{ij}dX_i dX_j \approx 2\varepsilon_{ij}dx_i dx_j \ .$$

1)

$$dx_1 = 1, dx_2 = dx_3 = 0$$

$$ds^2 - dS^2 = 2\varepsilon_{11}dx_1^2 = 4.10^{-3}$$

$$dS = dx_1 = 1 \Rightarrow ds^2 = 1 + 4.10^{-3} \Rightarrow ds = 1.002$$

$$\Rightarrow ds - dS = 1.002 - 1 = 0.002$$

2)

$$dx_2 = 1, dx_1 = dx_3 = 0$$

$$ds^2 - dS^2 = 2\varepsilon_{22} dx_2^2 = 4.10^{-3}$$

$$dS = dx_2 = 1 \Rightarrow ds^2 = 1 + 4.10^{-3} \Rightarrow ds = 1.002$$

$$\Rightarrow ds - dS = 1.002 - 1 = 0.002$$

3)

$$dx_1 = dx_2 = 1 \cdot \frac{\sqrt{2}}{2}, dx_3 = 0$$

$$\begin{aligned} ds^2 - dS^2 &= 2\varepsilon_{11} dx_1^2 + 2\varepsilon_{22} dx_2^2 + 2 \cdot 2\varepsilon_{12} dx_1 dx{+}_2 \\ &= (2 \cdot 2 + 2 \cdot 2 + 2 \cdot 2 \cdot 1) \cdot 10^{-3} \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} = 6 \cdot 10^{-3} \end{aligned}$$

$$dS = 1 \Rightarrow ds^2 = 1 + 6 \cdot 10^{-3} \Rightarrow ds = 1.003$$

$$\Rightarrow ds - dS = 1.003 - 1 = 0.003$$

We may also use (**B**2.153) to get

$$\varepsilon_N = \frac{ds - dS}{dS} = 0.003 \Rightarrow ds - dS = 0.003 \cdot dS = 0.003$$

Solution 2.17

Let $PQ = dS$ and $pq = ds$ the lengths given on figure 2.3. By (**B**2.151), one has

$$ds^2 = dS^2 + 2\varepsilon_{ij} dx_i dx_j \ .$$

In two dimensions, one writes

$$ds^2 = dS^2 + 2\varepsilon_{11} dx_1^2 + 2\varepsilon_{22} dx_2^2 + 4\varepsilon_{12} dx_1 dx_2 \ .$$

This last relation yields

$$\frac{ds^2 - dS^2}{dS^2} = 2\varepsilon_{11} \frac{dx_1^2}{dS^2} + 2\varepsilon_{22} \frac{dx_2^2}{dS^2} + 4\varepsilon_{12} \frac{dx_1 dx_2}{dS^2} \ . \tag{2.6}$$

Let us set

$$\varepsilon_N = \frac{ds - dS}{dS} \ .$$

The left hand side of (2.6) becomes

$$\begin{aligned} \frac{ds^2 - dS^2}{dS^2} &= \frac{ds^2}{dS^2} - 1 = \left(\frac{ds}{dS} - 1 + 1 \right)^2 - 1 = \left(\frac{ds - dS}{dS} + 1 \right)^2 - 1 \\ &= (\varepsilon_N + 1)^2 - 1 = \varepsilon_N^2 + 2\varepsilon_N \ . \end{aligned} \tag{2.7}$$

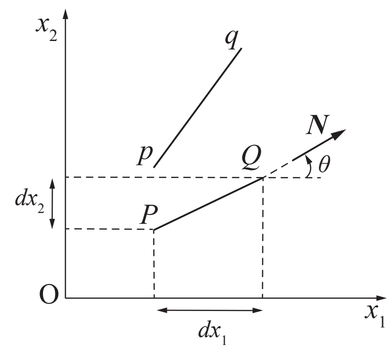

Figure 2.3 Deformation of a linear element.

In the case of small deformation, we allow that

$$\varepsilon_N^2 \to 0 \ . \tag{2.8}$$

Referring to figure 2.3, we define

$$\cos\theta = \frac{dx_1}{dS} \quad \text{and} \quad \sin\theta = \frac{dx_2}{dS} \ . \tag{2.9}$$

Combining (2.6) and (2.7) and using (2.8) and (2.9), we eventually obtain

$$\varepsilon_N = \varepsilon_{11}\cos^2\theta + \varepsilon_{22}\sin^2\theta + 2\varepsilon_{12}\cos\theta\sin\theta \ .$$

Utilizing trigonometric identities the last expression becomes

$$\varepsilon_N = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \frac{\varepsilon_{11}-\varepsilon_{22}}{2}\cos 2\theta + 2\varepsilon_{12}\sin 2\theta \ .$$

In a direction normal to $\boldsymbol{N}$, one has

$$\varepsilon_{N+\pi/2} = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \frac{\varepsilon_{11}-\varepsilon_{22}}{2}\cos 2(\pi/2+\theta) + 2\varepsilon_{12}\sin 2(\pi/2+\theta)$$

$$\varepsilon_{N+\pi/2} = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} - \frac{\varepsilon_{11}-\varepsilon_{22}}{2}\cos 2\theta - 2\varepsilon_{12}\sin 2\theta \ .$$

Chapitre 3

Dynamics of continuous media

Solution 3.1

Incompressibility (**B**3.45) requires $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$. Applying $\partial/\partial x_i$ to the velocity field, one has

$$\begin{aligned} \frac{\partial v_i}{\partial x_i} &= \frac{Ar^3\delta_{ii} - 3Ar^2 x_i \frac{\partial r}{\partial x_i}}{r^6} \\ &= \frac{Ar^3\delta_{ii} - 3Ar^2 x_i \frac{x_i}{r}}{r^6} \\ &= 0 \end{aligned}$$

since $x_i x_i = r^2$.

It is also possible to solve the problem by computing

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= \frac{\partial v_i}{\partial x_i} = A\frac{\partial}{\partial x_i}\left(x_i(x_j x_j)^{-3/2}\right) \\ &= A\left(\delta_{ii}(x_j x_j)^{-3/2} - \frac{3}{2}x_i \cdot 2(x_j x_j)^{-5/2} x_j \delta_{ji}\right) \\ &= A\left(\delta_{ii}(x_j x_j)^{-3/2} - \frac{3}{2}x_j \cdot 2(x_j x_j)^{-5/2} x_j\right) = 0 \ , \end{aligned}$$

as $\delta_{ii} = 3$.

Solution 3.2

The mass conservation equation (**B**3.53) gives

$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{v} = -\rho \frac{\partial v_i}{\partial x_i} = -\rho\left(\frac{3}{1+t}\right) \ .$$

Integrating from the initial time $t = 0$ till the present time t, one obtains

$$\int_{\rho_0}^{\rho} \frac{d\rho'}{\rho'} = -3\int_0^t \frac{dt'}{1+t'} \ ,$$

giving then

$$\frac{\rho}{\rho_0} = \frac{1}{(1+t)^3} \ . \tag{3.1}$$

Let us calculate the trajectoires linked to the velocity field (**B**2.35). One has

$$dx_i = v_i dt = \frac{x_i}{(1+t)} dt \ .$$

Integrating this relation from the initial time $t = 0$ till the actual time t

$$\int_{X_i}^{x_i} \frac{dx_i'}{x_i'} = \int_0^t \frac{dt'}{1+t'} \ ,$$

yields

$$\ln \frac{x_i}{X_i} = \ln(1+t)$$

and thus

$$\begin{aligned} x_1 = X_1(1+t),\ x_2 &= X_2(1+t),\ x_3 = X_3(1+t) \\ x_1 x_2 x_3 &= X_1 X_2 X_3 (1+t)^3 \ . \end{aligned} \tag{3.2}$$

The combination of (3.1) and (3.2) results in the relation $\rho\, x_1 x_2 x_3 = \rho_0 X_1 X_2 X_3$.

Solution 3.3

The incompressibility equation in cylindrical coordinates (**B**A.2) is written as

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \ .$$

It is easily deduced that the given velocity field is incompressible.

Solution 3.4

1) With the matrix $[\sigma]$

$$[\sigma] = \begin{pmatrix} 0 & Cx_1 & 0 \\ Cx_1 & 0 & -Cx_2 \\ 0 & -Cx_2 & 0 \end{pmatrix}$$

the static equilibrium equation (**B**3.126) gives

$$\begin{aligned} f_1 &= -\sigma_{1j,j} = 0 \\ f_2 &= -\sigma_{2j,j} = -C \\ f_3 &= -\sigma_{3j,j} = C \ . \end{aligned}$$

2) The normal vector to the plane given by surface $f(x_i) = 0$, at point P of coordinates $\boldsymbol{x}_P = (4, -4, 7)$ is defined by its gradient

$$\begin{aligned}
\boldsymbol{n}(\boldsymbol{x}_P) &= \frac{\nabla f(\boldsymbol{x}_P)}{\|\nabla f(\boldsymbol{x}_P)\|} \\
&= (2, 2, -1)^T \frac{1}{(2^2 + 2^2 + 1)^{\frac{1}{2}}} = \\
&= \frac{1}{3}(2, 2, -1)^T \ .
\end{aligned}$$

For the plane, the stress vector at point P becomes

$$\begin{aligned}
[t]_{plane} &= [\sigma][n] = \frac{1}{3} \begin{pmatrix} 0 & 4C & 0 \\ 4C & 0 & 4C \\ 0 & 4C & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} 8C \\ 8C - 4C \\ 8C \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 8C \\ 4C \\ 8C \end{pmatrix} \ .
\end{aligned}$$

The normal vector on the sphere at point P is

$$\begin{aligned}
\boldsymbol{n}(\boldsymbol{x}_P) &= \frac{\nabla f(\boldsymbol{x}_P)}{\|\nabla f(\boldsymbol{x}_P)\|} \\
&= (2x_1, 2x_2, 2x_3)^T \frac{1}{((2x_1)^2 + (2x_2)^2 + (2x_3)^2)^{\frac{1}{2}}} = \\
&= (x_1, x_2, x_3)^T \frac{1}{((x_1)^2 + (x_2)^2 + (x_3)^2)^{\frac{1}{2}}} = \\
&= (4, -4, 7)^T \frac{1}{(16 + 16 + 49)^{\frac{1}{2}}} = \\
&= \frac{1}{9}(4, -4, 7)^T \ .
\end{aligned}$$

The stress vector on the sphere at point P is

$$\begin{aligned}
[t]_S &= [\sigma][n] = \frac{1}{9} \begin{pmatrix} 0 & 4C & 0 \\ 4C & 0 & 4C \\ 0 & 4C & 0 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \\ 7 \end{pmatrix} \\
&= \frac{1}{9} \begin{pmatrix} -16C \\ 16C + 28C \\ -16C \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -16C \\ 44C \\ -16C \end{pmatrix} \ .
\end{aligned}$$

3) the principal stresses at point P are the eigenvalues of tensor $\boldsymbol{\sigma}(P)$ obtained solving (**B**3.111) :

$$\det(\boldsymbol{\sigma}(P) - \lambda \boldsymbol{I}) = 0 \ ,$$

where $\lambda = \sigma$.
With the definition of invariants (**B**1.121), one obtains

$$\begin{pmatrix} 0 & 4C & 0 \\ 4C & 0 & 4C \\ 0 & 4C & 0 \end{pmatrix}, \ I_1 = 0, \ I_2 = -32C^2, \ I_3 = 0$$

with

$$I_2 = -4 \cdot 4C^2 - 4 \cdot 4C^2 = -32C^2 \ .$$

The characteristic equation (**B**1.123) written for matrix $[\sigma]$ becomes

$$\lambda^3 - 32C^2\lambda = 0 \ ,$$

or

$$\lambda \left(\lambda^2 - 32C^2\right) = 0 \ .$$

The first solution is $\lambda = \sigma_2 = 0$. The other solutions are given by

$$\lambda^2 = 32C^2 \ ,$$

or

$$\sigma_{1,3} = \pm \left|C\right| \sqrt{32} \ .$$

For simplicity, the convention $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ is adopted.
The maximum shear stress is : $[\boldsymbol{\tau} \cdot \boldsymbol{\sigma n}]_{\max} = (\sigma_1 - \sigma_3)/2 = C\sqrt{32}$, where $\boldsymbol{\tau}$ is the tangent vector in P.
The deviatoric part of $\boldsymbol{\sigma}$ is by definition $\sigma^d_{ij} = \sigma_{ij} - (\sigma_{kk}/3)\delta_{ij}$. As in our case, the trace of σ is zero, the given stress tensor is identical to the deviatoric tensor and the principal deviatoric stresses are equal to the principal stresses.

Solution 3.5

In absence of body forces, the equilibrium equation is written as

$$\sigma_{ij,j} = 0 \ .$$

When applied to (**B**3.166), one has

$$\begin{gathered} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 8x_1 + 8x_2 - 8x_1 - 8x_2 = 0 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = -\frac{x_1}{2} - 8x_2 + \frac{x_1}{2} + 8x_2 = 0 \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0 \ . \end{gathered}$$

The given stress field satisfies equilibrium.

Solution 3.6

The body $\mathcal{B}$ is in equilibrium if the total force and the total moment are both equal to zero.

Equilibrium of forces

We write the equation with pressure P in the component F_1 of the force. Then, we use the divergence theorem to obtain (as $P = cst$)

$$F_1 = -\int_{\partial\omega} (Pn_1 + 0n_2 + 0n_3) ds = -\int_{\omega} \frac{\partial P}{\partial x_1} dv = 0 \ .$$

Similarly, one obtains $F_2 = F_3 = 0$. Thus, the equilibrium of forces is satisfied.

Equilibrium of moments

The moment with respect to the origin O of the force generated by the pressure, at point $\boldsymbol{x}$ is

$$\boldsymbol{M}(O) = \int_{\partial\omega} \boldsymbol{OM} \times (-P\boldsymbol{n}) ds = -P \int_{\partial\omega} \boldsymbol{OM} \times \boldsymbol{n} ds$$

$$\begin{aligned} \boldsymbol{OM} \times \boldsymbol{n} = & \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \\ & = \boldsymbol{e}_1(x_2n_3 - x_3n_2) - \boldsymbol{e}_2(x_1n_3 - x_3n_1) + \boldsymbol{e}_3(x_1n_2 - x_2n_1) \ . \end{aligned}$$

The first component of moment is expressed as

$$\begin{aligned} M_1(O) &= -P\boldsymbol{e}_1 \int_{\partial\omega} (x_2n_3 - x_3n_2) ds \\ &= -P\boldsymbol{e}_1 \int_{\partial\omega} \begin{pmatrix} 0 & -x_3 & x_2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} ds \ . \end{aligned}$$

Appying the divergence theorem to this last expression, one obtains

$$\begin{aligned} M_1(O) &= -P\boldsymbol{e}_1 \int_{\partial\omega} \begin{pmatrix} 0 & -x_3 & x_2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} ds \\ &= -P\boldsymbol{e}_1 \int_{\omega} \left(\frac{\partial 0}{\partial x_1} - \frac{\partial x_3}{\partial x_2} + \frac{\partial x_2}{\partial x_3} \right) ds = 0 \ . \end{aligned}$$

Similarly, one obtains $M_2(O) = M_3(O) = 0$.

Thus the moment is also equal to zero. The solid body is in equilibrium.

Solution 3.7

Equation (**B**3.111) gives

$$\det([\sigma] - \lambda[I]) = 0$$

Applied to (**B**3.167), one obtains

$$\left| \begin{pmatrix} p-\lambda & p & p \\ p & p-\lambda & p \\ p & p & p-\lambda \end{pmatrix} \right| = 0$$

or via (**B**1.120)

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0\ . \tag{3.3}$$

The invariants (**B**1.121) are

$$I_1 = 3p$$

$$I_2 = \begin{vmatrix} p & p \\ p & p \end{vmatrix} + \begin{vmatrix} p & p \\ p & p \end{vmatrix} + \begin{vmatrix} p & p \\ p & p \end{vmatrix} = 0$$

$$I_3 = \begin{vmatrix} p & p & p \\ p & p & p \\ p & p & p \end{vmatrix} = 0$$

Equation (3.3) becomes

$$-\lambda^3 + 3p\lambda^2 = 0$$

yielding

$$\begin{cases} \sigma_1 = 3p \\ \sigma_2 = 0 \\ \sigma_3 = 0 \end{cases}$$

The resulting stress state is one of uniform traction (p is supposed such that $p > 0$).

Applying (**B**3.111) to (**B**3.168), one has

$$\begin{vmatrix} p-\lambda & p & p \\ p & p-\lambda & p \\ p & p & -2p-\lambda \end{vmatrix} = 0$$

The invariants are $I_1 = 0, I_2 = -6p^2, I_3 = 0$. Equation (3.3) gives

$$\lambda(6p^2 - \lambda^2) = 0$$

One finds

$$\begin{cases} \sigma_1 = \sqrt{6}p \\ \sigma_2 = 0 \\ \sigma_3 = -\sqrt{6}p \end{cases}$$

This is a simple shear stress state because the two principal stresses are equal and opposite.

Applying (**B**3.111) to (**B**3.169), one has

$$\begin{vmatrix} 0-\lambda & p & p \\ p & 0-\lambda & p \\ p & p & 0-\lambda \end{vmatrix} = 0$$

The invariants are

$$I_1 = 0$$

$$I_2 = \begin{vmatrix} 0 & p \\ p & 0 \end{vmatrix} + \begin{vmatrix} 0 & p \\ p & 0 \end{vmatrix} + \begin{vmatrix} 0 & p \\ p & 0 \end{vmatrix} = -3p^2$$

$$I_3 = \begin{vmatrix} 0 & p & p \\ p & 0 & p \\ p & p & 0 \end{vmatrix} = 2p^3$$

Equation (3.3) becomes

$$-\lambda^3 + 3p^2\lambda + 2p^3 = 0$$

that can be decomposed as

$$(\lambda - 2p)(\lambda + p)^2 = 0$$

The principal stresses are

$$\begin{cases} \sigma_1 = 2p \\ \sigma_2 = \sigma_3 = -p \end{cases}$$

The resulting stress state is a three-dimensional stress state.

Solution 3.8

By the deviatoric tensor definition (**B**3.123), one has

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} = \sigma_{ij} - \frac{1}{3}\delta_{ij}I_1(\boldsymbol{\sigma}) \ . \tag{3.4}$$

The characteristic equation is given by (**B**1.123)

$$\boldsymbol{s}^3 - I_1(\boldsymbol{s})\boldsymbol{s}^2 + I_2(\boldsymbol{s})\boldsymbol{s} - I_3(\boldsymbol{s}) = 0 \ .$$

With (3.4), one has for the first invariant

$$I_1(\boldsymbol{s}) = s_{ii} = \sigma_{ii} - \frac{1}{3}\delta_{ii}\sigma_{kk} = \sigma_{ii} - \frac{1}{3}3\sigma_{kk} = 0 \ .$$

Thus

$$\boldsymbol{s}^3 + I_2(\boldsymbol{s})\boldsymbol{s} - I_3(\boldsymbol{s}) = 0 \ .$$

In the literature, the next form is used

$$\boldsymbol{s}^3 - I_2(\boldsymbol{s})\boldsymbol{s} - I_3(\boldsymbol{s}) = 0$$

with

$$I_2(\boldsymbol{s}) = -\frac{1}{2}\left(s_{ii}s_{jj} - s_{ij}s_{ji}\right), \quad I_3(\boldsymbol{s}) = \det \boldsymbol{s} \ .$$

The first relation is none other than (**B**3.171).

One writes successively

$$\begin{aligned} I_2(\boldsymbol{s}) &= \frac{1}{2}s_{ij}s_{ji} = \frac{1}{2}\left(\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}\right)\left(\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}\right) \\ &= \frac{1}{2}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{ij}\sigma_{kk} - \frac{1}{3}\delta_{ij}\sigma_{ij}\sigma_{kk} + \frac{1}{9}\delta_{ij}\delta_{ij}\sigma_{nn}\sigma_{kk}\right) \\ &= \frac{1}{2}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}\sigma_{jj}\sigma_{kk} - \frac{1}{3}\sigma_{jj}\sigma_{kk} + \frac{1}{3}\sigma_{nn}\sigma_{kk}\right) \\ &= \frac{1}{2}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}\sigma_{jj}\sigma_{kk}\right) \ . \end{aligned}$$

Using (**B**3.116) to replace $\sigma_{ij}\sigma_{ij}$ we obtain

$$\begin{aligned} I_2(\boldsymbol{s}) &= \frac{1}{2}\left(-2I_2(\boldsymbol{\sigma}) + I_1^2(\boldsymbol{\sigma}) - \frac{1}{3}I_1^2(\boldsymbol{\sigma})\right) = +\frac{1}{2}\left(-2I_2(\boldsymbol{\sigma}) + \frac{2}{3}I_1^2(\boldsymbol{\sigma})\right) \\ &= \frac{1}{3}I_1^2(\boldsymbol{\sigma}) - I_2(\boldsymbol{\sigma}) \ . \end{aligned} \qquad (\mathbf{B}3.171)$$

For the third invariant, we proceed as follows (cf. (**B**3.118))

$$\begin{aligned} I_3(\boldsymbol{s}) &= s_1s_2s_3 = \left(\sigma_1 - \frac{1}{3}I_1(\boldsymbol{\sigma})\right)\left(\sigma_2 - \frac{1}{3}I_1(\boldsymbol{\sigma})\right)\left(\sigma_3 - \frac{1}{3}I_1(\boldsymbol{\sigma})\right) \\ &= \sigma_1\sigma_2\sigma_3 - \frac{1}{3}I_1(\boldsymbol{\sigma})\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \\ &+ \frac{1}{9}I_1^2(\boldsymbol{\sigma})\left(\sigma_1 + \sigma_2 + \sigma_3\right) - \frac{1}{27}I_1^3(\boldsymbol{\sigma}) \\ &= I_3(\boldsymbol{\sigma}) - \frac{1}{3}I_1(\boldsymbol{\sigma})I_2(\boldsymbol{\sigma}) + \frac{1}{9}I_1^3(\boldsymbol{\sigma}) - \frac{1}{27}I_1^3(\boldsymbol{\sigma}) \\ &= \frac{2}{27}I_1^3(\boldsymbol{\sigma}) - \frac{1}{3}I_1(\boldsymbol{\sigma})I_2(\boldsymbol{\sigma}) + I_3(\boldsymbol{\sigma}) \ . \end{aligned} \qquad (\mathbf{B}3.172)$$

Solution 3.9

Through the divergence theorem the surface integral is converted into a volume integral

$$\int_{\partial\omega} P_{ij\ldots}\sigma_{pq}n_q\, ds = \int_{\omega} \frac{\partial(P_{ij\ldots}\sigma_{pq})}{\partial x_q} dv$$
$$= \int_{\omega} [\sigma_{pq}P_{ij\ldots,q} + P_{ij\ldots}\sigma_{pq,q}]dv \ .$$

Replacing $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}$ in this last relation by its expression produced by the momentum conservation law (**B**3.96), one obtains (**B**3.173).

Solution 3.10

The first Piola-Kirchhoff stress tensor is defined by the relation (**B**3.141) :

$$\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T} \ .$$

Multiplying (**B**3.141) right by $\boldsymbol{F}^T$, one has

$$\boldsymbol{P}\boldsymbol{F}^T = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}\boldsymbol{F}^T = J\boldsymbol{\sigma} \ .$$

The transpose of (**B**3.141) gives

$$\boldsymbol{P}^T = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}^T \ .$$

Multiplying left this last equation by $\boldsymbol{F}$, one finds

$$\boldsymbol{F}\boldsymbol{P}^T = J\boldsymbol{F}\boldsymbol{F}^{-1}\boldsymbol{\sigma}^T = J\boldsymbol{\sigma}^T = J\boldsymbol{\sigma}$$

and thus

$$\boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T \ . \qquad (\mathbf{B}3.144)$$

Solution 3.11

Equations (**B**3.149) and (**B**2.205) give

$$\boldsymbol{P}^* = \boldsymbol{Q}\boldsymbol{P};\ \ \boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F} \ .$$

Equation (**B**3.152) yields

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P} \Rightarrow \boldsymbol{P} = \boldsymbol{F}\boldsymbol{S} \ .$$

Therefore, one has

$$\boldsymbol{P}^* = \boldsymbol{F}^*\boldsymbol{S}^* \Rightarrow \boldsymbol{Q}\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{F}\boldsymbol{S}^* \Rightarrow \boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{P} = \boldsymbol{P} = \boldsymbol{F}\boldsymbol{S}^* \ .$$

However, by definition,

$$\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S} \ .$$

Consequently,

$$\boldsymbol{F}\boldsymbol{S} = \boldsymbol{F}\boldsymbol{S}^{*} \ ,$$

that leads to the result

$$\boldsymbol{S} = \boldsymbol{S}^{*} \ .$$

For the symmetry case, one has successively

$$\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} \Rightarrow \boldsymbol{S}^{T} = J(\boldsymbol{F}^{-T})^{T}\boldsymbol{\sigma}^{T}(\boldsymbol{F}^{-1})^{T} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}^{T}\boldsymbol{F}^{-T}$$

and since

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{T} \Rightarrow \boldsymbol{S} = \boldsymbol{S}^{T} \ .$$

Chapitre 4

Energetics

Solution 4.1

With Reynolds transport theorem (**B**3.23) and the continuity equation (**B**3.41), one has

$$\begin{aligned}\frac{d}{dt}\int_w \rho Q dv &= \int_w \left(\frac{D(\rho Q)}{Dt} + \rho Q \frac{\partial v_m}{\partial x_m}\right) dv \\ &= \int_w \left(\frac{D\rho}{Dt}Q + \rho\frac{DQ}{Dt} + \rho Q \frac{\partial v_m}{\partial x_m}\right) dv \\ &= \int_w \left(\rho\frac{DQ}{Dt} + Q\left(\frac{D\rho}{Dt} + \rho\frac{\partial v_m}{\partial x_m}\right)\right) dv \\ &= \int_w \rho \frac{dQ}{Dt} dv \ .\end{aligned}$$

Let us calculate the material derivative of the kinetic energy E_k

$$\frac{DE_k}{Dt} = \frac{d}{dt}\int_\omega \rho\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}\right) dv = \int_\omega \rho\frac{D}{Dt}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}\right) dv = \int_\omega \rho\, \boldsymbol{a}\cdot\boldsymbol{v}\, dv \ .$$

Replacing $\rho\boldsymbol{a}$ by the expression from (**B**3.96) one obtains equation (**B**4.26).

Solution 4.2

By the definition of the kinetic energy (**B**4.1) and the internal energy (**B**4.2), one has

$$\frac{D}{Dt}(E_k(t) + E_{int}(t)) = \frac{D}{Dt}\int_\omega \rho\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2} + u\right) dv \ .$$

Applying Reynolds theorem (**B**3.23) we are left with the following development

$$\begin{aligned}
&\frac{D}{Dt}\int_\omega \rho\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}+u\right)dv = \\
&\int_\omega [\frac{D}{Dt}\left(\rho\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}+u\right)\right)+\rho\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}+u\right)\boldsymbol{\nabla}\cdot\boldsymbol{v}]dv \\
=&\int_\omega [\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}+u\right)\left(\frac{D\rho}{Dt}+\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}\right)+\rho\frac{D}{Dt}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}+u\right)]dv\ .
\end{aligned}$$

The expression in the second term between parentheses of the last relation is the mass conservation (**B**3.41). As a consequence, this term vanishes. One finds

$$\frac{D}{Dt}\int_\omega \rho\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}+u\right)dv = \int_\omega \rho\left(\boldsymbol{v}\cdot\frac{D\boldsymbol{v}}{Dt}+\frac{Du}{Dt}\right)dv\ .$$

As $D\boldsymbol{v}/Dt$ is equal to $\boldsymbol{a}$, the problem is solved.

If we keep the material derivative of $\boldsymbol{v}$, we may write

$$\begin{aligned}
\int_\omega \rho\left(\boldsymbol{v}\cdot\frac{D\boldsymbol{v}}{Dt}+\frac{Du}{Dt}\right)dv &= \int_\omega \rho\left(\frac{D(\boldsymbol{v}\cdot\boldsymbol{v})}{2\,Dt}+\frac{Du}{Dt}\right)dv \\
&= \int_\omega \rho\frac{D}{Dt}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}+u\right)dv\ .
\end{aligned}$$

This shows that the material derivative of the integral over a material volume of a quantity equal to ρ times an expression is written in general form as the integral over a material volume of ρ times the material derivative of this expression. This statement constitutes a general theorem.

Solution 4.3

1) Inequation (**B**4.81) is given by

$$\frac{d}{dt}\int_\omega \rho s dv \geq \int_\omega \frac{r}{T}dv - \int_{\partial\omega}\frac{\boldsymbol{q}\cdot\boldsymbol{n}}{T}ds\ .$$

Using the result of the previous exercise, the left hand side becomes

$$\frac{d}{dt}\int_\omega \rho s dv = \int_\omega \rho\frac{Ds}{Dt}dv\ .$$

By the divergence theorem (**B**1.228), the surface integral of (**B**4.81) becomes

$$\int_{\partial\omega}\frac{q_k n_k}{T}ds = \int_\omega \frac{\partial}{\partial x_k}\left(\frac{q_k}{T}\right)dv\ .$$

Applying the localisation principle one obtains

$$\rho \frac{Ds}{Dt} \geq \frac{r}{T} - \operatorname{div}\left(\frac{\boldsymbol{q}}{T}\right) . \tag{4.1}$$

2) With the equations (**B**4.23) and (**B**4.25), one has

$$\rho \frac{Du}{Dt} - \boldsymbol{\sigma} : \boldsymbol{d} + \operatorname{div} \boldsymbol{q} = r . \tag{4.2}$$

Employing the index notation, it is easily shown that

$$\operatorname{div}\left(\frac{\boldsymbol{q}}{T}\right) = \frac{1}{T} \operatorname{div} \boldsymbol{q} - \frac{\boldsymbol{q} \cdot \boldsymbol{\nabla} T}{T^2} . \tag{4.3}$$

Bringing (4.2) and (4.3) in (4.1), we find Clausius-Duhem inequality

$$\rho \frac{Ds}{Dt} \geq \frac{1}{T}\left(\rho \frac{Du}{Dt} - \boldsymbol{\sigma} : \boldsymbol{d}\right) + \frac{1}{T^2} \boldsymbol{q} \cdot \boldsymbol{\nabla} T . \tag{B4.83}$$

3) If we introduce the Helmholtz specific free energy ,

$$f = u - Ts , \tag{B4.84}$$

the Clausius-Duhem inequality (**B**4.83) is rewritten as

$$\rho \frac{Df}{Dt} \leq tr(\boldsymbol{\sigma} \boldsymbol{L}) - \rho s \frac{DT}{Dt} - \frac{\boldsymbol{q} \cdot \boldsymbol{\nabla} T}{T} . \tag{B4.85}$$

Solution 4.4

1) The principle of internal energy conservation is given by the relations (**B**4.23) and (**B**4.25)

$$\rho \frac{Du}{Dt} = \boldsymbol{\sigma} : \boldsymbol{d} - \operatorname{div} \boldsymbol{q} + r . \tag{4.4}$$

For a perfect fluid, the term $\boldsymbol{\sigma} : \boldsymbol{d}$ becomes $-p\, tr \boldsymbol{d} = -p \boldsymbol{\nabla} \cdot \boldsymbol{v}$.

2) Substituting the enthalpy definition in (4.4), one has

$$\rho \frac{Dh}{Dt} - \rho \frac{D}{Dt}\left(\frac{p}{\rho}\right) = -p \frac{\partial v_i}{\partial x_i} - \frac{\partial q_i}{\partial x_i} + r . \tag{4.5}$$

The development of the term $\frac{D}{Dt}\left(\frac{p}{\rho}\right)$ gives

$$\frac{D}{Dt}\left(\frac{p}{\rho}\right) = \frac{1}{\rho} \frac{Dp}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt} .$$

Substituting this result in (4.5), one finds

$$\begin{aligned}\rho\frac{Dh}{Dt} &= \frac{Dp}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} - p\frac{\partial v_i}{\partial x_i} - \frac{\partial q_i}{\partial x_i} + r \\ &= \frac{Dp}{Dt} - \frac{p}{\rho}\left(\frac{D\rho}{Dt} + \rho\frac{\partial v_i}{\partial x_i}\right) - \frac{\partial q_i}{\partial x_i} + r \\ &= \frac{Dp}{Dt} - \frac{\partial q_i}{\partial x_i} + r\ ,\end{aligned}$$

where we have used the mass conservation law (**B**3.41).

3) If moreover, the flow is adiabatic, i. e. $\boldsymbol{q} = \boldsymbol{0}$ and $r = 0$, one has

$$\rho\,\frac{Dh}{Dt} = \frac{Dp}{Dt}\ .$$

Solution 4.5

As rigid body rotation implies by (**B**4.61) $\boldsymbol{c} = \boldsymbol{0}$, it follows that $\dot{\boldsymbol{c}} = \boldsymbol{0}$. By (**B**2.211) and taking (**B**4.63) into account one has in

$$\boldsymbol{v}^* = \dot{\boldsymbol{c}} + \dot{\boldsymbol{Q}}\boldsymbol{x} + \boldsymbol{Q}\boldsymbol{v} = \boldsymbol{Q}\boldsymbol{v} + \boldsymbol{\Omega}\boldsymbol{x}\ .$$

With (**B**2.60), one finds

$$\boldsymbol{v}^* = \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}\ . \tag{4.6}$$

The vector $\boldsymbol{\omega}$ is the dual vector of $\boldsymbol{\Omega}$, that expresses the angular velocity of rigid body rotation. We rewrite (**B**4.48)

$$\begin{aligned}\rho\dot{u} - \boldsymbol{\sigma} : \boldsymbol{\nabla}\boldsymbol{v} + \operatorname{div}\boldsymbol{q} - r \,+\, \boldsymbol{v}\cdot(\rho\boldsymbol{a} - \boldsymbol{div}\,\boldsymbol{\sigma} - \rho\boldsymbol{b}) \\ +\left(\frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + u\right)(\dot{\rho} + \rho\operatorname{div}\boldsymbol{v}) = 0\ ,\end{aligned} \tag{4.7}$$

(where the notation $\dot{\rho}$ designates the material derivative in Lagrangian description), with starred quantities. We first replace $\boldsymbol{v}^*$ by its value (4.6) and from the resulting equation, we substract (4.7). Using the relations (**B**4.49)-(**B**4.53), (**B**2.212), (**B**2.213), one obtains

$$\begin{aligned}-\boldsymbol{\sigma} : \boldsymbol{\Omega} \,+\, &\frac{(\boldsymbol{\omega}\times\boldsymbol{x})\cdot(\boldsymbol{\omega}\times\boldsymbol{x})}{2}(\dot{\rho} + \rho\operatorname{div}\boldsymbol{v}) + (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\boldsymbol{v}(\dot{\rho} + \rho\operatorname{div}\boldsymbol{v}) \\ &+\, (\boldsymbol{\omega}\times\boldsymbol{x})\cdot(\rho\boldsymbol{a} - \boldsymbol{div}\,\boldsymbol{\sigma} - \rho\boldsymbol{b}) = 0\ .\end{aligned}$$

This must be true for any rigid body rotation and thus we deduce the equations of mass and momentum conservation. The remaining term $\boldsymbol{\sigma} : \boldsymbol{\Omega}$ must vanish. Due to the antisymmetric character of $\boldsymbol{\Omega}$, this imposes the symmetry of $\boldsymbol{\sigma}$.

Chapitre 5

Constitutive equations : basic principles

Solution 5.1

The vector field $\boldsymbol{u}$ is spatially objective and satisfies relation (**B**2.197)

$$\boldsymbol{u}^* = \boldsymbol{Q}\boldsymbol{u} \ .$$

One has on the other hand that

$$(\nabla \boldsymbol{u})^*_{ij} = \frac{\partial u^*_i}{\partial x^*_j} = \frac{\partial u^*_i}{\partial x_k}\frac{\partial x_k}{\partial x^*_j} \ .$$

By (**B**2.195), one finds

$$\frac{\partial x^*_j}{\partial x_k} = Q_{jk}$$

and its inverse $\partial x_k / \partial x^*_j$ is $Q^{-1}_{kj} = Q^T_{kj}$. Thus one has

$$(\nabla \boldsymbol{u})^*_{ij} = \frac{\partial u^*_i}{\partial x^*_j} = Q_{il}\frac{\partial u_l}{\partial x_k}Q^T_{kj}$$

or

$$(\nabla \boldsymbol{u})^* = \boldsymbol{Q} \, \nabla \boldsymbol{u} \, \boldsymbol{Q}^T \ . \qquad (\mathbf{B}5.61)$$

Solution 5.2

Through the relation (**B**2.213) and definitions (**B**2.181) and (**B**2.183) of $\boldsymbol{d}$ and $\dot{\boldsymbol{\omega}}$, respectively, and taking into account equation (**B**2.56)

$$\dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{Q}}^T = 0 \ ,$$

one writes successively

$$\begin{aligned}
(\boldsymbol{L^*})^T &= \boldsymbol{Q}(\boldsymbol{L}^T\boldsymbol{Q}^T + \dot{\boldsymbol{Q}}^T), \\
\boldsymbol{d}^* &= \frac{1}{2}(\boldsymbol{L}^* + (\boldsymbol{L^*})^T) = \boldsymbol{Q}\frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T)\boldsymbol{Q}^T + \frac{1}{2}(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{Q}}^T) \\
&= \boldsymbol{Q}\boldsymbol{d}\boldsymbol{Q}^T, \\
\dot{\boldsymbol{\omega}}^* &= \frac{1}{2}(\boldsymbol{L}^* - (\boldsymbol{L^*})^T) = \boldsymbol{Q}\frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T)\boldsymbol{Q}^T + \frac{1}{2}(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T - \boldsymbol{Q}\dot{\boldsymbol{Q}}^T) \\
&= \boldsymbol{Q}\dot{\boldsymbol{\omega}}\boldsymbol{Q}^T + \dot{\boldsymbol{Q}}\boldsymbol{Q}^T \ .
\end{aligned}$$

Tensor $\boldsymbol{d}$ is spatially objective and tensor $\dot{\boldsymbol{\omega}}$ is not. Indeed, if two observers record the rotation rate of a continuous media, their observations differ by a quantity equal to their relative rotation rate.

Solution 5.3

Note first that $D/Dt^* = D/Dt$. Tensor $\boldsymbol{T}$ being spatially objective, if we take the material derivative, one obtains successively

$$\frac{D\boldsymbol{T}^*}{Dt} = \frac{D(\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T)}{Dt} = \frac{D\boldsymbol{Q}}{Dt}\boldsymbol{T}\boldsymbol{Q}^T + \boldsymbol{Q}\frac{D\boldsymbol{T}}{Dt}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{T}\frac{D\boldsymbol{Q}^T}{Dt} \ .$$

This shows that the material derivative of a second order tensor is not spatially objective.

Solution 5.4

With the help of results of problems 5.2 and 5.3, and relation (**B**2.56), we write

$$\begin{aligned}
\dot{\boldsymbol{T}}^* + \boldsymbol{T}^*\dot{\boldsymbol{\omega}}^* - \dot{\boldsymbol{\omega}}^*\boldsymbol{T}^* &= \dot{\boldsymbol{Q}}\boldsymbol{T}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{T}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{T}\dot{\boldsymbol{Q}}^T \\
+\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T(\boldsymbol{Q}\dot{\boldsymbol{\omega}}\boldsymbol{Q}^T + \dot{\boldsymbol{Q}}\boldsymbol{Q}^T) &- (\boldsymbol{Q}\dot{\boldsymbol{\omega}}\boldsymbol{Q}^T + \dot{\boldsymbol{Q}}\boldsymbol{Q}^T)\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T \\
&= \boldsymbol{Q}\dot{\boldsymbol{T}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{T}\dot{\boldsymbol{\omega}}\boldsymbol{Q}^T - \boldsymbol{Q}\dot{\boldsymbol{\omega}}\boldsymbol{T}\boldsymbol{Q}^T \\
&= \boldsymbol{Q}(\dot{\boldsymbol{T}} + \boldsymbol{T}\dot{\boldsymbol{\omega}} - \dot{\boldsymbol{\omega}}\boldsymbol{T})\boldsymbol{Q}^T \ .
\end{aligned}$$

This shows that relation (**B**5.62) is spatially objective.

Solution 5.5

Equation (**B**5.64) is nothing else than problem 5.3 solved for $\boldsymbol{T} = \boldsymbol{d}$. Therefore, taking (**B**2.56) into account, from (**B**2.213) and its transpose, one has to within a factor 2

$$\begin{aligned}
\dot{\boldsymbol{d}}^* + \boldsymbol{d}^*\boldsymbol{L}^* + \boldsymbol{L}^{T*}\boldsymbol{d}^* &= \dot{\boldsymbol{Q}}\boldsymbol{d}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{d}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{d}\dot{\boldsymbol{Q}}^T \\
&+ \boldsymbol{Q}\boldsymbol{d}\boldsymbol{Q}^T(\boldsymbol{Q}\boldsymbol{L} + \dot{\boldsymbol{Q}})\boldsymbol{Q}^T + \boldsymbol{Q}(\boldsymbol{L}^T\boldsymbol{Q}^T + \dot{\boldsymbol{Q}}^T)\boldsymbol{Q}\boldsymbol{d}\boldsymbol{Q}^T \\
&= \boldsymbol{Q}\dot{\boldsymbol{d}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{d}\boldsymbol{L}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{L}^T\boldsymbol{d}\boldsymbol{Q}^T \ .
\end{aligned}$$

Chapitre 6

Classical Constitutive Equations

Solution 6.1

Equation (**B**2.88) gives

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} \ ,$$

from which we obtain via (**B**2.179)

$$\dot{\boldsymbol{C}} = \dot{\boldsymbol{F}}^T \boldsymbol{F} + \boldsymbol{F}^T \dot{\boldsymbol{F}} = \boldsymbol{F}^T \boldsymbol{L}^T \boldsymbol{F} + \boldsymbol{F}^T \boldsymbol{L} \boldsymbol{F} \ .$$

As for the simple fluid $\boldsymbol{F} = \boldsymbol{I}$, one finds

$$\dot{\boldsymbol{C}} = \boldsymbol{L}^T + \boldsymbol{L} = 2\boldsymbol{d} \ .$$

Solution 6.2

Equation (**B**4.23) is

$$\rho \frac{Du}{Dt} = \boldsymbol{\sigma} : (\boldsymbol{\nabla v}) - \operatorname{div} \boldsymbol{q} + r \ .$$

The term $\boldsymbol{\sigma} : (\boldsymbol{\nabla v})$ with the constitutive equation (**B**6.14) becomes with the help of (**B**4.25)

$$\boldsymbol{\sigma} : \boldsymbol{L} = \boldsymbol{\sigma} : \boldsymbol{d} = -p \, tr \, \boldsymbol{d} + \lambda (tr \, \boldsymbol{d})^2 + 2\mu (\boldsymbol{d} : \boldsymbol{d}) \ .$$

Thus, for the Newtonian viscous fluid, one obtains

$$\rho \frac{Du}{Dt} = -p \, tr \, \boldsymbol{d} + \lambda (tr \, \boldsymbol{d})^2 + 2\mu (\boldsymbol{d} : \boldsymbol{d}) - \operatorname{div} \boldsymbol{q} + r \ . \qquad (6.1)$$

The perfect fluid is inviscid, i.e. $\lambda = \mu = 0$. One finds

$$\rho \frac{Du}{Dt} = -p \, tr \, \boldsymbol{d} - \operatorname{div} \boldsymbol{q} + r \ .$$

If the perfect fluid is an ideal gas, then its internal energy is given by relation (**B**6.143) and the previous equation becomes

$$\rho c_v \frac{DT}{Dt} = -p \, tr \, \boldsymbol{d} - \operatorname{div} \boldsymbol{q} + r \ .$$

Solution 6.3

The Newtonian viscous incompressible fluid satisfies the constraint div $\boldsymbol{v} = tr\, \boldsymbol{d} = 0$. In this case, relation (6.1) yields

$$\rho \frac{Du}{Dt} = 2\mu(\boldsymbol{d} : \boldsymbol{d}) - \operatorname{div} \boldsymbol{q} + r \; .$$

For the perfect fluid, one finds

$$\rho \frac{Du}{Dt} = -\operatorname{div} \boldsymbol{q} + r \; .$$

Solution 6.4

1) Relation (**B**2.108) shows that $\boldsymbol{U}$ has $\boldsymbol{A}_i$ as eigenvectors. By (**B**2.109), one writes the spectral decomposition

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \boldsymbol{A}_i \otimes \boldsymbol{A}_i \; .$$

With (**B**2.88), one has successively

$$\begin{aligned} \boldsymbol{C} = \boldsymbol{U}\boldsymbol{U} &= \sum_{i=1}^{3} \sum_{j=1}^{3} \lambda_i (\boldsymbol{A}_i \otimes \boldsymbol{A}_i) \lambda_j (\boldsymbol{A}_j \otimes \boldsymbol{A}_j) = \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \lambda_i \lambda_j (\boldsymbol{A}_i \otimes \boldsymbol{A}_i)(\boldsymbol{A}_j \otimes \boldsymbol{A}_j) = \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \lambda_i \lambda_j \delta_{ij} (\boldsymbol{A}_i \otimes \boldsymbol{A}_i) = \\ &= \sum_{i=1}^{3} \lambda_i^2 (\boldsymbol{A}_i \otimes \boldsymbol{A}_i) \; . \end{aligned}$$

The isotropic hyperelastic material has for constitutive equation (**B**6.51)

$$\boldsymbol{S} = 2 \frac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}} \; .$$

With (**B**6.64), one obtains relation (**B**6.67) for $\partial \widehat{\mathcal{W}}(\boldsymbol{C}) / \partial \boldsymbol{C}$ and finally, one has for (**B**6.68)

$$\boldsymbol{S} = \sum_{i=1}^{3} \frac{1}{\lambda_i} \frac{\partial \phi}{\partial \lambda_i} \boldsymbol{A}_i \otimes \boldsymbol{A}_i \, ,$$

which shows that $\boldsymbol{S}$ has $\boldsymbol{A}_i$ as eigenvectors.

2) Relation (**B**2.111) shows that $\boldsymbol{V}$ has $\boldsymbol{b}_i = \boldsymbol{R}\boldsymbol{A}_i$ as eigenvectors. By (**B**2.113), one writes the spectral decomposition

$$\boldsymbol{V} = \sum_{i=1}^{3} \lambda_i \boldsymbol{b}_i \otimes \boldsymbol{b}_i \ .$$

With(**B**2.89), one has successively

$$\begin{aligned}
\boldsymbol{c} = \boldsymbol{V}\boldsymbol{V} &= \sum_{i=1}^{3}\sum_{j=1}^{3} \lambda_i(\boldsymbol{b}_i \otimes \boldsymbol{b}_i)\lambda_j(\boldsymbol{b}_j \otimes \boldsymbol{b}_j) = \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} \lambda_i\lambda_j(\boldsymbol{b}_i \otimes \boldsymbol{b}_i)(\boldsymbol{b}_j \otimes \boldsymbol{b}_j) = \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} \lambda_i\lambda_j\delta_{ij}(\boldsymbol{b}_i \otimes \boldsymbol{b}_i) = \\
&= \sum_{i=1}^{3} \lambda_i^2(\boldsymbol{b}_i \otimes \boldsymbol{b}_i) \ .
\end{aligned}$$

This proves that $\boldsymbol{c}$ has $\boldsymbol{b}_i$ as eigenvectors.
In section **B**6.5.1, after long developments, one obtains relation (**B**6.72)

$$\boldsymbol{\sigma} = J^{-1}\left(\sum_{i=1}^{3} \lambda_i \frac{\partial \phi}{\partial \lambda_i}\, \boldsymbol{b}_i \otimes \boldsymbol{b}_i\right) \ ,$$

which shows that $\boldsymbol{\sigma}$ has $\boldsymbol{b}_i$ as eigenvectors.

Solution 6.5

Relation (**B**6.61) multiplied left by $\boldsymbol{F}$ and right by $\boldsymbol{F}^T$ gives

$$\frac{1}{2}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T = I_3\frac{\partial \Phi}{\partial I_3}\boldsymbol{F}\boldsymbol{C}^{-1}\boldsymbol{F}^T + \left(\frac{\partial \Phi}{\partial I_1} + I_1\frac{\partial \Phi}{\partial I_2}\right)\boldsymbol{F}\boldsymbol{I}\boldsymbol{F}^T - \frac{\partial \Phi}{\partial I_2}\boldsymbol{F}\boldsymbol{C}\boldsymbol{F}^T \ . \tag{6.2}$$

With the help of (**B**2.88) and (**B**2.89), one has

$$\boldsymbol{F}\boldsymbol{C}^{-1}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{F}^{-1}\boldsymbol{F}^{-T}\boldsymbol{F}^T = \boldsymbol{I}$$

and

$$\boldsymbol{F}\boldsymbol{C}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{c}\boldsymbol{c} = \boldsymbol{c}^2 \ .$$

Equation (6.2) taking (**B**3.152) into account becomes (**B**6.63)

$$\boldsymbol{\sigma} = 2J^{-1}\left(I_3(\boldsymbol{c})\frac{\partial \Phi}{\partial I_3(\boldsymbol{c})}\boldsymbol{I} + \left(\frac{\partial \Phi}{\partial I_1(\boldsymbol{c})} + I_1(\boldsymbol{c})\frac{\partial \Phi}{\partial I_2(\boldsymbol{c})}\right)\boldsymbol{c} - \frac{\partial \Phi}{\partial I_2(\boldsymbol{c})}\boldsymbol{c}^2\right) \ .$$

Solution 6.6

Relation (**B**6.61) is

$$\boldsymbol{S} = 2\left(I_3 \frac{\partial \Phi}{\partial I_3} \boldsymbol{C}^{-1} + \left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2} \right) \boldsymbol{I} - \frac{\partial \Phi}{\partial I_2} \boldsymbol{C} \right) \ .$$

The Cayley-Hamilton equation (**B**1.123) applied to $\boldsymbol{C}$ gives

$$\boldsymbol{C}^3 - I_1 \boldsymbol{C}^2 + I_2 \boldsymbol{C} - I_3 \boldsymbol{I} = \boldsymbol{0}$$

and thus

$$I_3 \boldsymbol{C}^{-1} = \boldsymbol{C}^2 - I_1 \boldsymbol{C} + I_2 \boldsymbol{I} \ .$$

Combining it with (**B**6.61) one finds

$$\begin{aligned} \frac{\boldsymbol{S}}{2} &= \frac{\partial \Phi}{\partial I_3} \left(\boldsymbol{C}^2 - I_1 \boldsymbol{C} + I_2 \boldsymbol{I} \right) + \left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2} \right) \boldsymbol{I} - \frac{\partial \Phi}{\partial I_2} \boldsymbol{C} \\ &= \left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2} + I_2 \frac{\partial \Phi}{\partial I_3} \right) \boldsymbol{I} - \left(I_1 \frac{\partial \Phi}{\partial I_3} + \frac{\partial \Phi}{\partial I_2} \right) \boldsymbol{C} + \frac{\partial \Phi}{\partial I_3} \boldsymbol{C}^2 \ . \end{aligned}$$

Relation (**B**6.63) is given by

$$\boldsymbol{\sigma} = 2J^{-1} \left(I_3 \frac{\partial \Phi}{\partial I_3} \boldsymbol{I} + \left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2} \right) \boldsymbol{c} - \frac{\partial \Phi}{\partial I_2} \boldsymbol{c}^2 \right) \ .$$

The Cayley-Hamilton equation (**B**1.123) applied to $\boldsymbol{c}$ gives

$$\boldsymbol{c}^3 - I_1 \boldsymbol{c}^2 + I_2 \boldsymbol{c} - I_3 \boldsymbol{I} = \boldsymbol{0}$$

and thus

$$\boldsymbol{c}^2 = I_1 \boldsymbol{c} - I_2 \boldsymbol{I} + I_3 \boldsymbol{c}^{-1} \ .$$

Combining it with (**B**6.63) one obtains

$$\begin{aligned} \frac{\boldsymbol{\sigma}}{2J^{-1}} &= I_3 \frac{\partial \Phi}{\partial I_3} \boldsymbol{I} + \left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2} \right) \boldsymbol{c} - I_1 \frac{\partial \Phi}{\partial I_2} \boldsymbol{c} + I_2 \frac{\partial \Phi}{\partial I_2} \boldsymbol{I} - I_3 \frac{\partial \Phi}{\partial I_2} \boldsymbol{c}^{-1} \\ \boldsymbol{\sigma} &= 2J^{-1} \left(\left(I_2 \frac{\partial \Phi}{\partial I_2} + I_3 \frac{\partial \Phi}{\partial I_3} \right) \boldsymbol{I} + \frac{\partial \Phi}{\partial I_1} \boldsymbol{c} - I_3 \frac{\partial \Phi}{\partial I_2} \boldsymbol{c}^{-1} \right) \ . \end{aligned}$$

Solution 6.7

According to (**B**6.80), we can write

$$\begin{aligned} \Phi(I_1, I_2, I_3) \ &= C_{000} + C_{100}(I_1 - 3) + C_{010}(I_2 - 3) + C_{001}(I_3 - 1) \\ &+ \ C_{111}(I_1 - 3)(I_2 - 3)(I_3 - 1) + \ldots \ . \end{aligned}$$

In the reference configuration, one has

$$\lambda_1 = \lambda_2 = \lambda_3 = 1 \text{ or } \boldsymbol{C} = \boldsymbol{I}$$

and then

$$I_1 = 3,\ I_2 = 3,\ I_3 = 1\ .$$

Consequently, one obtains

$$\Phi(3, 3, 1) = C_{000}\ .$$

If $C_{000} = 0$, then the energy is zero, at the reference configuration.

The partial derivatives of Φ with respect to the invariants are

$$\frac{\partial \Phi}{\partial I_1} = C_{100} + C_{111}(I_2 - 3)(I_3 - 1) + \ldots$$

$$\frac{\partial \Phi}{\partial I_2} = C_{010} + C_{111}(I_1 - 3)(I_3 - 1) + \ldots$$

$$\frac{\partial \Phi}{\partial I_3} = C_{001} + C_{111}(I_1 - 3)(I_2 - 3) + \ldots$$

Thus, at the reference configuration (**B**6.62) becomes

$$\frac{\partial \Phi}{\partial I_1} + 2\frac{\partial \Phi}{\partial I_2} + \frac{\partial \Phi}{\partial I_3} = C_{100} + 2C_{010} + C_{001} = 0\ .$$

Solution 6.8

The first equality of (**B**6.59) gives with the help of (**B**1.144)

$$\begin{aligned}
\frac{\partial I_1}{\partial \boldsymbol{C}} &= \sum_{i=1}^{3} \frac{\partial(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}{\partial \lambda_i^2} (\boldsymbol{n}_i \otimes \boldsymbol{n}_i) \\
&= \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + \boldsymbol{n}_2 \otimes \boldsymbol{n}_2 + \boldsymbol{n}_3 \otimes \boldsymbol{n}_3 \\
&= \sum_1^3 \boldsymbol{n}_i \otimes \boldsymbol{n}_i \\
&= \boldsymbol{I}\ .
\end{aligned}$$

The second equality of (**B**6.59) gives with the help of (**B**1.144)

$$\begin{aligned}
\frac{\partial I_2}{\partial \boldsymbol{C}} &= \sum_{i=1}^{3} \frac{\partial(\lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2)}{\partial \lambda_i^2}(\boldsymbol{n}_i \otimes \boldsymbol{n}_i) \\
&= (\lambda_2^2 + \lambda_3^2)(\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) + (\lambda_3^2 + \lambda_1^2)(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2) + (\lambda_1^2 + \lambda_2^2)(\boldsymbol{n}_3 \otimes \boldsymbol{n}_3) \\
&= (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) + (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2) \\
&+ (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\boldsymbol{n}_3 \otimes \boldsymbol{n}_3) \\
&- \lambda_1^2(\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) - \lambda_2^2(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2) - \lambda_3^2(\boldsymbol{n}_3 \otimes \boldsymbol{n}_3) \\
&= I_1 \boldsymbol{I} - \boldsymbol{C} \ .
\end{aligned}$$

Solution 6.9

The pressure in the inflated balloon is given by equation (**B**6.102)

$$p_i(\lambda) = 4C_{10}\frac{e_i}{R}\frac{1}{\lambda}\left(1 - \frac{1}{\lambda^6}\right) \ .$$

The maximum pressure is obtained when the derivative of p with respect to the stretch ratio λ vanishes

$$\frac{\partial p_i(\lambda)}{\partial \lambda} = 0 \ .$$

One has

$$\frac{d}{d\lambda}\left(\frac{1}{\lambda} - \frac{1}{\lambda^7}\right) = -\frac{1}{\lambda^2} + \frac{7}{\lambda^8} = 0$$

and thus

$$\lambda^6 = 7 \Rightarrow \lambda = \sqrt[6]{7} = 1.383$$

$$p_i^{max} = 4C_{10}\frac{e_i}{R}\frac{1}{1.383}\left(1 - \frac{1}{7}\right) = 2.479\frac{C_{10}e_i}{R} \ .$$

Solution 6.10

The Ogden energy function is relation (**B**6.86)

$$\phi(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i}(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3) \ .$$

The principal stresses are given by relation (**B**6.78)

$$\sigma_k = -p + \lambda_k \frac{\partial \phi}{\partial \lambda_k}, \quad k = 1, 2, 3 \ .$$

Therefore, one writes

$$\sigma_1 = -p + \sum_{i=1}^{N} \mu_i \lambda_1^{\alpha_i}$$
$$\sigma_2 = -p + \sum_{i=1}^{N} \mu_i \lambda_2^{\alpha_i}$$
$$\sigma_3 = -p + \sum_{i=1}^{N} \mu_i \lambda_3^{\alpha_i} \ .$$

– Case of uniaxial stretch : $\sigma_1 = \sigma$, $\sigma_2 = \sigma_3 = 0$ and $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = \lambda^{-1/2}$ (incompressibility). One obtains due to these conditions :
$\sigma = -\sum_{i=1}^{N} \mu_i \lambda^{-\frac{\alpha_i}{2}} + \sum_{i=1}^{N} \mu_i \lambda^{\alpha_i} = \sum_{i=1}^{N} \mu_i (\lambda^{\alpha_i} - \lambda^{\frac{-\alpha_i}{2}})$
– Case of biaxial stretch : $\sigma_1, \sigma_2 \neq 0, \sigma_3 = 0$, $\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}$ (incompressibility). One obtains due to these conditions :
$\sigma_1 = -\sum_{i=1}^{N} \mu_i (\frac{1}{\lambda_1\lambda_2})^{\alpha_i} + \sum_{i=1}^{N} \mu_i \lambda_1^{\alpha_i} = \sum_{i=1}^{N} \mu_i (\lambda_1^{\alpha_i} - (\frac{1}{\lambda_1\lambda_2})^{\alpha_i})$
$\sigma_2 = -\sum_{i=1}^{N} \mu_i (\frac{1}{\lambda_1\lambda_2})^{\alpha_i} + \sum_{i=1}^{N} \mu_i \lambda_2^{\alpha_i} = \sum_{i=1}^{N} \mu_i (\lambda_2^{\alpha_i} - (\frac{1}{\lambda_1\lambda_2})^{\alpha_i})$
– Case of equibiaxial stretch : $\sigma_1 = \sigma_2 = \sigma$, $\sigma_3 = 0$ (particular case of biaxial stretch) and $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = \lambda^{-2}$ (by incompressibility). One obtains due to these conditions :
$\sigma = -\sum_{i=1}^{N} \mu_i \lambda^{-2\alpha_i} + \sum_{i=1}^{N} \mu_i \lambda^{\alpha_i} = \sum_{i=1}^{N} \mu_i (\lambda^{\alpha_i} - \lambda^{-2\alpha_i})$.
Applying the prescribed values of the terms, namely $N = 3, \alpha_1 = 1,3, \alpha_2 = 5, \alpha_3 = -2, \mu_1 = 0,63$MPa, $\mu_2 = 0,0012$MPa and $\mu_3 = -0,01$MPa, we can plot $\sigma_1, \sigma_2, \sigma_3$ as a function of their corresponding elongation.

Figure 6.1 shows the evolution of the uniaxial stress with respect to its elongation, while figure 6.2 exhibits the evolution of the equibiaxial stress.

Solution 6.11

The stress tensor has only one non zero component, namely $\sigma_{11} = S$, where S is the traction load per unit surface.

The free energy f is given by (**B**6.159)

$$\frac{1}{\rho}\ \sigma_{ij} = \frac{\partial f}{\partial \epsilon_{ij}}$$

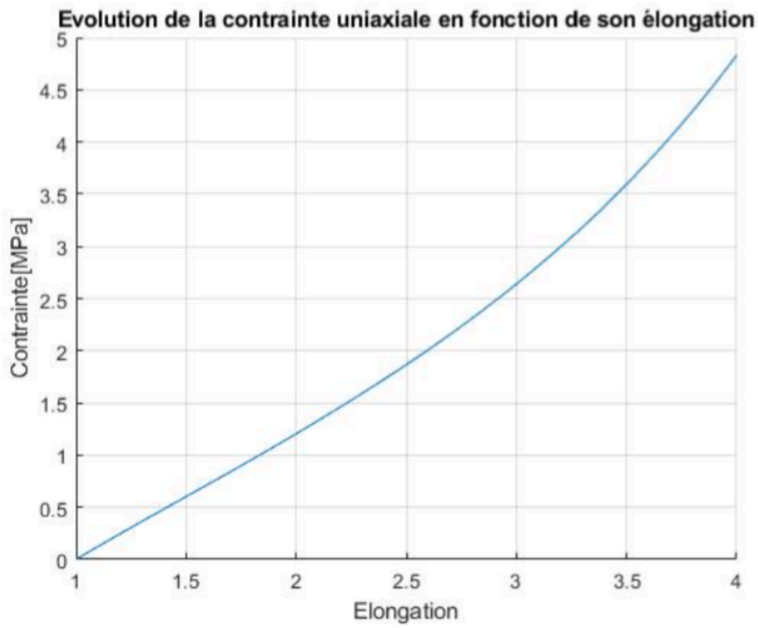

Figure 6.1 Uniaxial stress

and thus

$$\frac{S}{\rho} = \frac{\partial f}{\partial \epsilon_{11}} \ . \tag{6.3}$$

By (**B**6.110), one has

$$\epsilon_{11} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} \ S = \frac{1}{E} \ S \ ,$$

where E is the Young's modulus (**B**6.109). One writes (6.3) as follows

$$\frac{S}{\rho} = \frac{\partial f}{\partial S} \frac{\partial S}{\partial \epsilon_{11}} \ .$$

So

$$\frac{\partial f}{\partial S} = \frac{S}{\rho E}$$

and finally we have integrating :

$$f = \frac{1}{\rho E} \frac{S^2}{2} + f_0 \ ,$$

where f_0 is the unstressed free energy in natural state.

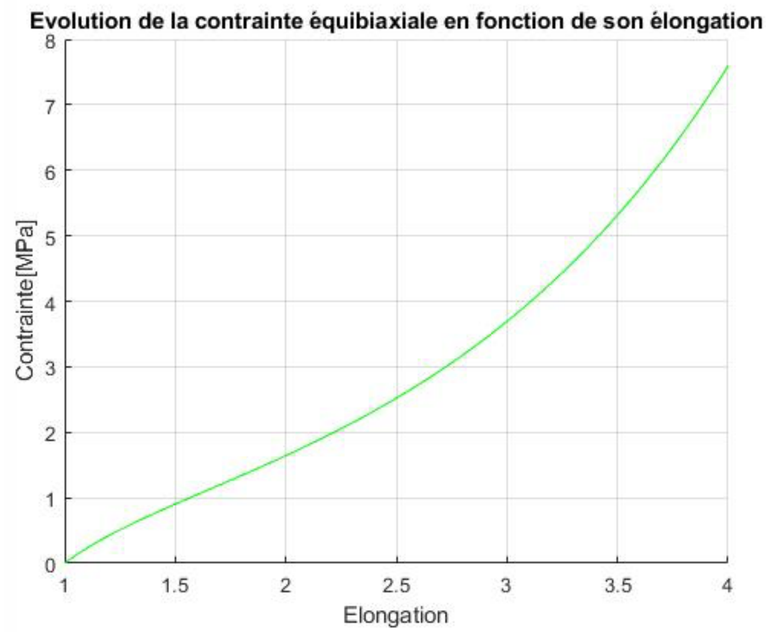

Figure 6.2 Equibiaxial stress

Solution 6.12

The development of $f(\boldsymbol{\epsilon}, T)$ in the neighborhood of $\boldsymbol{\epsilon} = \mathbf{0}, T = T_0$ is written as :

$$\begin{aligned}\rho\ f\ &= \rho\ f_0 - \rho\ s_0\ (T - T_0) + \frac{\lambda}{2}\epsilon_{ii}\epsilon_{kk} + \mu\epsilon_{ij}\epsilon_{ij}\\ &+\ \epsilon_{ij}c_{ij}(T - T_0) - \frac{\rho c}{2T_0}(T - T_0)^2\ ,\end{aligned} \tag{B6.165}$$

where we have eliminated all terms of order greater than 2, and where the coefficients f_0, s_0, c_{ij} and c are still to be determined. The factors ρ and $\frac{\rho}{T_0}$ were added to simplify subsequent steps.

For an isotropic material, c_{ij} must isotropic, of the form $a\ \delta_{ij}$ with a a scalar. Taking this scalar as $a = -(3\lambda + 2\mu)\alpha$, with α yet to be determined, one has

$$c_{ij}\ \epsilon_{ij}\ (T - T_0) = -(3\lambda + 2\mu)\ \alpha\ \epsilon_{kk}\ (T - T_0)\ . \tag{B6.166}$$

Furthermore, combining (**B**6.165) and (**B**6.166) one obtains by (**B**6.159) the next relation

$$\begin{aligned}\sigma_{ij}\ &= \rho\frac{\partial f}{\partial \epsilon_{ij}}\\ &= \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\delta_{ij}\ .\end{aligned}$$

Inverting to obtain $\boldsymbol{\epsilon}$ as a function of $\boldsymbol{\sigma}$, one has

$$\boldsymbol{\epsilon} = \frac{1}{2\mu} \left\{ \boldsymbol{\sigma} + \left[2\mu\alpha(T - T_0) - \frac{\lambda}{3\lambda + 2\mu} tr\boldsymbol{\sigma} \right] \boldsymbol{I} \right\} ,$$

where we have used the relation

$$\epsilon_{kk} = \frac{1}{3\lambda + 2\mu} \, \sigma_{kk} + 3\alpha(T - T_0) \, .$$

As a reminder, α is the thermal expansion coefficient and has dimensions of the inverse of temperature. If we consider the case of free dilatation without exterior stresses, then $\boldsymbol{\sigma} = \mathbf{0}$ and one has

$$\boldsymbol{\epsilon} = \alpha(T - T_0)\boldsymbol{I} \ .$$

As $T = T(x_1)$, ϵ_{11} is the only non zero component and consequently

$$\epsilon_{11} = \epsilon_{11}(x_1) = \alpha(T_1 - T_0) \ \frac{x_1}{L} \ .$$

Solution 6.13

To solve this problem,we will use the solution of the second part of exercise 6.2. Introducing the Fourier conduction law $\boldsymbol{q} = -k\,\boldsymbol{\nabla} T$ in (6.2), one obtains

$$\rho c_v \frac{DT}{Dt} = -p\, tr\, \boldsymbol{d} + \mathrm{div}(k\, \boldsymbol{\nabla} T) + r \ .$$

By the mass conservation equation (**B**3.41), one has the equality :

$$tr\, \boldsymbol{d} = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

and the energy equation becomes

$$\rho c_v \frac{DT}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt} + \mathrm{div}(k\, \boldsymbol{\nabla} T) + r \ .$$

Using the state equation (**B**6.136), we transform the previous equation in the relation :

$$\rho c_v \frac{DT}{Dt} = \frac{Dp}{Dt} - \rho R \frac{DT}{Dt} + \mathrm{div}(k\, \boldsymbol{\nabla} T) + r \ .$$

Finally, one can write taking (**B**6.141) into account

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \mathrm{div}(k\, \boldsymbol{\nabla} T) + r \ .$$

Solution 6.14

1) The relations (**B**6.175)-(**B**6.176) lead to write

$$\sigma_{ij}^d + \sigma_0 \delta_{ij} = 3\lambda \varepsilon_0 \delta_{ij} + 2\mu(\varepsilon_{ij}^d + \varepsilon_0 \delta_{ij}) \ . \tag{6.4}$$

Let us recall that the deviatoric tensors have a zero trace

$$tr\, \sigma_{ij}^d = tr\, \varepsilon_{ij}^d = 0 \ .$$

Therefore, computing the trace of (6.4), one obtains

$$3\sigma_0 = 3\lambda\varepsilon_0.3 + 2\mu\varepsilon_0.3$$

and

$$\sigma_0 = 3\lambda\varepsilon_0 + 2\mu\varepsilon_0 = (3\lambda + 2\mu)\varepsilon_0 \ .$$

The definition (**B**6.119)

$$K = \frac{3\lambda + 2\mu}{3}$$

gives

$$\sigma_0 = 3K\varepsilon_0 \ .$$

We rewrite (6.4) successively

$$\begin{aligned} \sigma_{ij}^d + 3K\varepsilon_0\delta_{ij} &= 3\lambda\varepsilon_0\delta_{ij} + 2\mu\varepsilon_{ij}^d + 2\mu\varepsilon_0\delta_{ij} \\ &= (3\lambda + 2\mu)\varepsilon_0\delta_{ij} + 2\mu\varepsilon_{ij}^d \\ &= 3K\varepsilon_0\delta_{ij} + 2\mu\varepsilon_{ij}^d \ . \end{aligned}$$

One finds

$$\sigma_{ij}^d = 2\mu\varepsilon_{ij}^d \ .$$

2) Let us recall that for a second order symmetric tensor $\boldsymbol{L}$, one has $\boldsymbol{L}\boldsymbol{n} = \lambda\boldsymbol{n}$, where λ is the eigenvalue of $\boldsymbol{L}$ and $\boldsymbol{n}$ the corresponding eigenvector (sec. 1.3.8).
For the deviatoric stress tensor σ_{ij}^d, one has

$$\sigma_{ij}^d n_j = \lambda n_i \ . \tag{6.5}$$

We modify (6.5) as follows

$$\sigma_{ij}^d n_j + \sigma_0 n_i = \sigma_0 n_i + \lambda n_i = (\lambda + \sigma_0) n_i$$

With the help of (**B**6.175), one writes

$$\sigma_{ij}^d n_j + \sigma_0 n_i = \sigma_{ij}^d n_j + \sigma_0 \delta_{ij} n_j = (\sigma_{ij}^d + \sigma_0\delta_{ij}) n_j = \sigma_{ij} n_j \ .$$

And we obtain $\sigma_{ij}n_j = (\lambda + \sigma_0)n_i$. This shows that σ^d_{ij} and σ_{ij} have the same eigenvectors.
As regards the displacements, we proceed in a similar fashion. Using (**B**6.177) in (6.5) gives

$$\varepsilon^d_{ij}n_j = \frac{\lambda}{2\mu}n_i \ . \tag{6.6}$$

This shows that σ^d_{ij} and ε^d_{ij} have the same eigenvectors and consequently, the same principal directions. Using (**B**6.176) to rewrite (6.6) leads to the relation

$$(\varepsilon_{ij} - \varepsilon_0\delta_{ij})n_j = \frac{\lambda}{2\mu}n_i$$

or

$$\varepsilon_{ij}n_j = (\varepsilon_0 + \frac{\lambda}{2\mu})n_i \ .$$

Comparing this last relation with (6.6), one concludes that ε_{ij} and ε^d_{ij} have the same eigenvectors. Finally, as
σ^d_{ij} and σ_{ij} have the same eigenvectors n_i,
σ^d_{ij} and ε^d_{ij} have the same eigenvectors n_i,
ε_{ij} and ε^d_{ij} have the same eigenvectors n_i,
we conclude that ε_{ij} and σ_{ij} have the same eigenvectors n_i and consequently, the same principal directions.

3) The potential strain energy is defined by the next relation

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\varepsilon_{ij}\sigma_{ij} \ .$$

One thus has

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\varepsilon_{ij}\sigma_{ij} = \frac{1}{2}\varepsilon_{ij}(\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}) = \frac{1}{2}\lambda\varepsilon^2_{kk} + \mu\varepsilon_{ij}\varepsilon_{ij} \ .$$

With the help of (**B**6.176), one writes

$$\begin{aligned} W(\boldsymbol{\varepsilon}) &= \frac{1}{2}\lambda(3\varepsilon_0)^2 + \mu(\varepsilon^d_{ij} + \varepsilon_0\delta_{ij})(\varepsilon^d_{ij} + \varepsilon_0\delta_{ij}) \\ &= \frac{9}{2}\lambda(\varepsilon_0)^2 + \mu(\varepsilon^d_{ij}\varepsilon^d_{ij} + 3\varepsilon_0^2) \\ &= \frac{9}{2}\lambda(\varepsilon_0)^2 + 3\mu(\varepsilon_0)^2 + \mu\varepsilon^d_{ij}\varepsilon^d_{ij} \\ &= \frac{9}{2}\frac{3\lambda + 2\mu}{3}\varepsilon_0^2 + \mu\varepsilon^d_{ij}\varepsilon^d_{ij} \\ &= \frac{9}{2}K\varepsilon_0^2 + \mu\varepsilon^d_{ij}\varepsilon^d_{ij} \ . \end{aligned} \qquad (\mathbf{B}6.178)$$

4) For the stability condition

$$W(\boldsymbol{\varepsilon}) > 0$$

to be satisfied, as relation (**B**6.178) is composed of two squares, the coefficients must be such that

$$K > 0 \quad \text{et} \quad \mu > 0\ .$$

Solution 6.15

1) With (**B**6.106),(**B**6.180) and (**B**6.181), one writes

$$\begin{aligned}
\varepsilon_{ij} &= -\frac{\lambda\delta_{ij}}{2\mu(3\lambda+2\mu)}\sigma_{nn} + \frac{\sigma_{ij}}{2\mu} \\
&= -\frac{\lambda\delta_{ij}}{2\mu(3\lambda+2\mu)}\sigma_{nn} + \frac{\sigma\delta_{ij}}{2\mu} \\
&= \left(-\frac{3\lambda}{2\mu(3\lambda+2\mu)} + \frac{1}{2\mu}\right)\sigma\delta_{ij} \\
&= \frac{-3\lambda+3\lambda+2\mu}{2\mu(3\lambda+2\mu)}\sigma\delta_{ij} \\
&= \frac{\sigma}{3K}\delta_{ij} \\
&= \varepsilon\delta_{ij}\ .
\end{aligned}$$

2) By Hooke's law (**B**6.104) and (**B**6.182), one has

$$\begin{aligned}
\sigma_{ij} &= \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \\
&= \frac{\lambda\gamma}{2}(m_k n_k + m_k n_k)\delta_{ij} + \mu\gamma(m_i n_j + m_j n_i) \\
&= 0 + \mu\gamma(m_i n_j + m_j n_i) \\
&= \tau(m_i n_j + m_j n_i)\ .
\end{aligned}$$

3) With (**B**6.106) and (**B**6.184), one obtains

$$\begin{aligned}
\varepsilon_{ij} &= -\frac{\lambda\delta_{ij}}{2\mu(3\lambda+2\mu)}\sigma n_m n_m + \frac{1}{2\mu}\sigma n_i n_j \\
&= -\frac{\lambda\sigma}{2\mu(3\lambda+2\mu)}(\delta_{ij} - n_i n_j + n_i n_j) + \frac{1}{2\mu}\sigma n_i n_j \\
&= \left(\frac{-\lambda+3\lambda+2\mu}{2\mu(3\lambda+2\mu)}\right)\sigma n_i n_j - \frac{\lambda\sigma}{2\mu(3\lambda+2\mu)}(\delta_{ij} - n_i n_j) \\
&= \left(\frac{(\lambda+\mu)\sigma}{\mu(3\lambda+2\mu)}\right) n_i n_j - \frac{\lambda\sigma}{2\mu(3\lambda+2\mu)}(\delta_{ij} - n_i n_j) \\
&= \varepsilon_n n_i n_j + \varepsilon_T(\delta_{ij} - n_i n_j)\ .
\end{aligned}$$

CHAPITRE 7

Introduction to Solid Mechanics

Solution 7.1

With (**B**7.18), equations (**B**7.21)-(**B**7.23) yield

$$
\begin{aligned}
\sigma_{11} &= \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{\partial u_1}{\partial x_1}(1-\nu)+\nu\frac{\partial u_2}{\partial x_2}\right)\\
\sigma_{22} &= \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{\partial u_2}{\partial x_2}(1-\nu)+\nu\frac{\partial u_1}{\partial x_1}\right)\\
\sigma_{12} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)\ .
\end{aligned}
$$

Let us evaluate the partial derivatives of the components of the stress tensor with respect to the space variables

$$
\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1} &= \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{\partial^2 u_1}{\partial x_1^2}(1-\nu)+\nu\frac{\partial^2 u_2}{\partial x_1\partial x_2}\right)\\
\frac{\partial \sigma_{22}}{\partial x_2} &= \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{\partial^2 u_2}{\partial x_2^2}(1-\nu)+\nu\frac{\partial^2 u_1}{\partial x_1\partial x_2}\right)\\
\frac{\partial \sigma_{12}}{\partial x_1} &= \frac{E}{2(1+\nu)}\left(\frac{\partial^2 u_1}{\partial x_1\partial x_2}+\frac{\partial^2 u_2}{\partial x_1^2}\right)\\
\frac{\partial \sigma_{12}}{\partial x_2} &= \frac{E}{2(1+\nu)}\left(\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_2}{\partial x_1\partial x_2}\right)\ .
\end{aligned}
$$

Inserting these derivatives in the equilibrium equations given by (**B**7.20)

$$
\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1}+\frac{\partial \sigma_{12}}{\partial x_2}+f_1=0\\
\frac{\partial \sigma_{22}}{\partial x_2}+\frac{\partial \sigma_{12}}{\partial x_1}+f_2=0\ ,
\end{aligned}
$$

one finds successively

$$
\begin{aligned}
&\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+f_1=0\\
&\frac{E}{(1+\nu)(1-2\nu)}\left(\frac{\partial^2 u_1}{\partial x_1^2}(1-\nu)+\nu\frac{\partial^2 u_2}{\partial x_1\partial x_2}\right)\\
&+\frac{E}{2(1+\nu)}\left(\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_2}{\partial x_1\partial x_2}\right)+f_1=0\\
&\frac{2}{1-2\nu}\left(\frac{\partial^2 u_1}{\partial x_1^2}(1-\nu)+\nu\frac{\partial^2 u_2}{\partial x_1\partial x_2}\right)+\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_2}{\partial x_1\partial x_2}+\frac{f_1}{\mu}=0\\
&\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 u_1}{\partial x_1^2}+\frac{2\nu}{1-2\nu}\frac{\partial^2 u_2}{\partial x_1\partial x_2}+\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_2}{\partial x_1\partial x_2}+\frac{f_1}{\mu}=0\\
&\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 u_1}{\partial x_1^2}+\frac{1}{1-2\nu}\frac{\partial^2 u_2}{\partial x_1\partial x_2}+\frac{\partial^2 u_1}{\partial x_2^2}+\frac{f_1}{\mu}=0\\
&\frac{1}{1-2\nu}\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_1^2}+\frac{1}{1-2\nu}\frac{\partial^2 u_2}{\partial x_1\partial x_2}+\frac{\partial^2 u_1}{\partial x_2^2}+\frac{f_1}{\mu}=0\\
&\mu\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}\right)+\frac{\mu}{1-2\nu}\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)+f_1=0\\
&\mu\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}\right)+(\lambda+\mu)\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)+f_1=0\ .
\end{aligned}
$$

A similar reasoning leads to (**B**7.313).

Solution 7.2

With (**B**7.41), the equations (**B**7.43) are written as

$$
\begin{aligned}
\sigma_{11}&=\frac{E}{1-\nu^2}\left(\frac{\partial u_1}{\partial x_1}+\nu\frac{\partial u_2}{\partial x_2}\right)\\
\sigma_{22}&=\frac{E}{1-\nu^2}\left(\frac{\partial u_2}{\partial x_2}+\nu\frac{\partial u_1}{\partial x_1}\right)\\
\sigma_{12}&=\frac{E}{2(1+\nu)}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)\ .
\end{aligned}
$$

Let us evaluate the partial derivatives of the components of the stress tensor with respect to the space variables

$$
\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1} &= \frac{E}{1-\nu^2}\left(\frac{\partial^2 u_1}{\partial x_1^2} + \nu \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) \\
\frac{\partial \sigma_{22}}{\partial x_2} &= \frac{E}{1-\nu^2}\left(\frac{\partial^2 u_2}{\partial x_2^2} + \nu \frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right) \\
\frac{\partial \sigma_{12}}{\partial x_1} &= \frac{E}{2(1+\nu)}\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2}\right) \\
\frac{\partial \sigma_{12}}{\partial x_2} &= \frac{E}{2(1+\nu)}\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) .
\end{aligned}
$$

The equilibrium equations are given by (**B**7.20)

$$
\begin{aligned}
&\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0 \\
&\frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{12}}{\partial x_1} + f_2 = 0 .
\end{aligned}
$$

Thus the first equilibrium equation becomes successively

$$
\begin{aligned}
&\frac{E}{1-\nu^2}\left(\frac{\partial^2 u_1}{\partial x_1^2} + \nu \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) + \frac{E}{2(1+\nu)}\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) + f_1 = 0 \\
&\frac{2}{1-\nu}\left(\frac{\partial^2 u_1}{\partial x_1^2} + \nu \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{f_1}{\mu} = 0 \\
&\frac{2}{1-\nu}\frac{\partial^2 u_1}{\partial x_1^2} + \frac{2\nu}{1-\nu}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{f_1}{\mu} = 0 \\
&\frac{2}{1-\nu}\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_1^2} - \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{1+\nu}{1-\nu}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{f_1}{\mu} = 0 \\
&\mu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2}\right) + \mu\frac{1+\nu}{1-\nu}\frac{\partial^2 u_1}{\partial x_1^2} + \mu\frac{1+\nu}{1-\nu}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + f_1 = 0 \\
&\mu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2}\right) + \frac{E}{2(1-\nu)}\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) + f_1 = 0 .
\end{aligned}
$$

A similar reasoning leads to (**B**7.315).

Solution 7.3

The Navier equations (**B**7.6) are written as

$$
(\lambda + \mu) u_{k,ki} + \mu u_{i,jj} + f_i = 0 . \tag{7.1}
$$

With the u_i expression given by (**B**7.315), one has successively

$$\mu u_{i,jj} = \frac{\lambda + 2\mu}{\lambda + \mu} g_{i,mmjj} - g_{n,nijj}$$

as well as

$$(\lambda + \mu) u_k = \frac{\lambda + 2\mu}{\mu} g_{k,mm} - \frac{\lambda + \mu}{\mu} g_{n,nk} \ .$$

Taking twice the derivative of this last relation, one obtains

$$\begin{aligned}(\lambda + \mu) u_{k,ki} &= \frac{\lambda + 2\mu}{\mu} g_{k,mmki} - \frac{\lambda + \mu}{\mu} g_{n,nkki} = \\ &= \frac{\lambda + 2\mu}{\mu} g_{k,kmmi} - \frac{\lambda + \mu}{\mu} g_{k,kmmi} \\ &= g_{k,kmmi} \ .\end{aligned}$$

Inserting these expressions in the Navier equation (7.1), one has

$$\begin{aligned}g_{k,kmmi} &+ \frac{\lambda + 2\mu}{\lambda + \mu} g_{i,mmjj} - g_{n,nijj} \\ &= g_{k,kmmi} + \frac{\lambda + 2\mu}{\lambda + \mu} g_{i,mmjj} - g_{k,kmmi} \\ &= \frac{\lambda + 2\mu}{\lambda + \mu} g_{i,mmjj} \\ &= 0\end{aligned}$$

if $g_{i,mmnn} = 0$.

Solution 7.4

1) We use identity (**B**1.238)

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}$$

that gives

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \nabla^2 \boldsymbol{u} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} \ .$$

We introduce this last relation in (**B**7.7) that becomes

$$\mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \left(\nabla^2 \boldsymbol{u} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} \right) = 0 \ .$$

Therefore

$$(\lambda + 2\mu) \nabla^2 \boldsymbol{u} + (\lambda + \mu) \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} = 0 \ .$$

2) From relation (**B**7.7), one writes

$$\frac{\mu}{\lambda+\mu}\nabla^2\boldsymbol{u} + \nabla(\boldsymbol{\nabla}\cdot\boldsymbol{u}) = 0\ .$$

The elasticity constants are linked together. By (**B**6.109), $2\nu = \lambda/(\lambda+\mu)$. Thus $\mu/(\lambda+\mu) = 1-2\nu$ and then the result.

3) We use relation (**B**1.238) in the Navier equation (**B**7.7)

$$\mu\left(\nabla(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u}\right) + (\lambda+\mu)\nabla(\boldsymbol{\nabla}\cdot\boldsymbol{u}) = 0$$

from which we get easily

$$(\lambda+2\mu)\nabla(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \mu\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u} = 0\ .$$

Solution 7.5

Taking the divergence of the relation

$$(\lambda+2\mu)\nabla^2\boldsymbol{u} = \rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}\ , \qquad (\mathbf{B}7.209)$$

one obtains

$$(\lambda+2\mu)\,\mathrm{div}\ \nabla^2\boldsymbol{u} = \rho\frac{\partial^2(\mathrm{div}\ \boldsymbol{u})}{\partial t^2}\ .$$

Afterwards we use (**B**1.191) to obtain

$$(\lambda+2\mu)\nabla^2(\mathrm{div}\ \boldsymbol{u}) = \rho\frac{\partial^2(\mathrm{div}\ \boldsymbol{u})}{\partial t^2}\ .$$

Solution 7.6

With div $\boldsymbol{u} = \varepsilon_{ii} = 0$, the motion equations (**B**7.202) become

$$\mu\,\nabla^2\boldsymbol{u} = \rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}\ .$$

We take the curl of this last relation

$$\mu\nabla\times\left(\nabla^2\boldsymbol{u}\right) = \rho\nabla\times\frac{\partial^2\boldsymbol{u}}{\partial t^2}\ .$$

Using (**B**1.237), one finds

$$\mu\nabla^2(\boldsymbol{\nabla}\times\boldsymbol{u}) = \rho\frac{\partial^2(\boldsymbol{\nabla}\times\boldsymbol{u})}{\partial t^2}\ . \qquad (\mathbf{B}7.319)$$

Solution 7.7

The function Φ is a solution of the problem if it satisfies the biharmonic equation (**B**7.38). To this end, let us first calculate its Laplacian. One finds (cf. (**B**A.27))

$$\begin{aligned}\nabla^2\Phi &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Phi \\ &= \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} \\ &= -\frac{4B}{r^2}\sin 2\theta\ .\end{aligned}$$

Afterwards, one evaluates the double Laplacian. One has

$$\begin{aligned}\nabla^4\Phi &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(-\frac{4B}{r^2}\sin 2\theta\right) \\ &= -\frac{24B}{r^4}\sin 2\theta + \frac{1}{r}\frac{8B}{r^3}\sin 2\theta + \frac{1}{r^2}\frac{4B}{r^2}\cdot 2\cdot 2\sin 2\theta \\ &= 0\ .\end{aligned}$$

The stress components are given by the relations (**B**A.28)-(**B**A.30)

$$\begin{aligned}\sigma_{rr} &= \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = -\frac{4B}{r^2}\sin 2\theta \\ \sigma_{\theta\theta} &= \frac{\partial^2\Phi}{\partial r^2} = 0 \\ \sigma_{r\theta} &= \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\partial r} = \frac{1}{r^2}(A + 2B\cos 2\theta)\ .\end{aligned}$$

The equilibrium of the corner is given by the relation

$$\int_{-\alpha}^{\alpha}(\sigma_{r\theta} r d\theta)r - M = 0$$

and thus

$$\int_{-\alpha}^{\alpha}(A + 2B\cos 2\theta)d\theta = M\ . \tag{7.2}$$

To the condition (7.2), we must add the one expressing that the corner edges are free and not subjected to any shear stress. One writes

$$\sigma_{r\theta}(\theta = \alpha) = A + 2B\cos 2\alpha = 0\ . \tag{7.3}$$

We thus have $A = -2B\cos 2\alpha$. Replacing A by this value in (7.2) and integrating, we find

$$B = \frac{M}{2\sin 2\alpha - 4\alpha\cos 2\alpha}\ .$$

The radial stress becomes

$$\sigma_{rr} = -\frac{4M}{2\sin 2\alpha - 4\alpha\cos 2\alpha}\frac{\sin 2\theta}{r^2} = -\frac{2C}{r^2}\sin 2\theta$$

with

$$C = -\frac{M}{\sin 2\alpha - 2\alpha\cos 2\alpha} \ .$$

The shear stress is

$$\sigma_{r\theta} = -\frac{M}{r^2}\frac{\cos 2\alpha - \cos 2\theta}{\sin 2\alpha - 2\alpha\cos 2\alpha} \ .$$

Solution 7.8

(a) One verifies easily that the stress function (**B**7.322) satisfies the biharmonic equation. From the expressions in the appendix **B**(A.28)-**B**(A.30), one obtains

$$\sigma_{rr} = \frac{2C\cos\theta}{r}, \quad \sigma_{\theta\theta} = \sigma_{r\theta} = 0 \ .$$

To find the C constant, one considers the equilibrium of the slab at a distance r from the origin. Thus

$$P = C\int_{-\alpha}^{\alpha}\frac{2\cos\theta}{r} r\cos\theta\, d\theta = 2C\int_{-\alpha}^{\alpha}\cos^2\theta\, d\theta \ .$$

Therefore

$$P = 2C\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_{-\alpha}^{\alpha} = C\left[2\alpha + \sin 2\alpha\right]$$

Thus, one finds

$$C = \frac{P}{2\alpha + \sin 2\alpha} \ .$$

The stress takes the form

$$\sigma_{rr} = \frac{2P\cos\theta}{r(2\alpha + \sin 2\alpha)}$$

(b) Setting $\alpha = \pi/2$, one obtains the stresses for a plate subject to a linear load

$$\sigma_{rr} = \frac{2P}{\pi}\frac{\cos\theta}{r}, \quad \sigma_{\theta\theta} = \sigma_{r\theta} = 0 \ .$$

Let us note that the difference of this result with (**B**7.163) comes from the fact that the directions of σ_{rr} are opposite in both problems.

Chapitre 8

Introduction to Newtonian Fluid Mechanics

Solution 8.1

The Navier-Stokes equations are treated in section **B**8.4. The mass conservation law is given by (**B**3.44) or (**B**8.9)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0 \ .$$

The momentum conservation equation (**B**3.96) gives

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho \frac{D v_i}{D t} \ .$$

The Newtonian viscous fluid constitutive equation is written as (**B**6.14)

$$\sigma_{ij} = -p\, \delta_{ij} + \lambda\, d_{kk}\, \delta_{ij} + 2\mu\, d_{ij} \ .$$

Inserting (**B**6.14) in (**B**3.96), one obtains

$$\rho \frac{D v_i}{D t} = \frac{\partial}{\partial x_j} \left(-p\delta_{ij} + \lambda d_{kk} \delta_{ij}\right) + \frac{\partial}{\partial x_j} (2\mu d_{ij}) + \rho b_i \ ,$$

and thus the relation (**B**8.10)

$$\rho \frac{D v_i}{D t} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i} (\lambda d_{kk}) + \frac{\partial}{\partial x_j} (2\mu d_{ij}) + \rho b_i \ ,$$

as

$$\frac{\partial}{\partial x_j} \left(-p + \lambda d_{kk}\right) \delta_{ij} = \frac{\partial}{\partial x_i} \left(-p + \lambda d_{kk}\right) \ .$$

The energy equation (**B**8.7), taking the Fourier conduction law into account (**B**6.123), becomes

$$\rho c_v \frac{DT}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt} + \lambda\,(tr\boldsymbol{d})^2 + 2\mu\,\boldsymbol{d} : \boldsymbol{d} + \mathrm{div}(k\,\boldsymbol{\nabla}T) + r\ .$$

If the coefficients λ and μ are constant, the momentum equation gives

$$\rho\frac{Dv_i}{Dt} = -\frac{\partial p}{\partial x_i} + \lambda\frac{\partial}{\partial x_i}(d_{kk}) + \mu\Delta v_i + \mu\frac{\partial}{\partial x_i}(\frac{\partial v_j}{\partial x_j}) + \rho b_i\ .$$

If the flow is incompressible, it results that div $\boldsymbol{v} = tr\,\boldsymbol{d} = 0$ and the previous equation simplifies itself to

$$\rho\frac{Dv_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\Delta v_i + \rho b_i\ .$$

Solution 8.2

The solutions obtained in section **B**8.7.1, i.e. (**B**8.51) and (**B**8.63), for the plane Couette and Poiseuille flows, respectively, result from linear differential equations. As the non-linear terms of the Navier-Stokes equations do not intervene in this problem, one invokes the principle of linear superposition and the solution of the combined plane Couette-Poiseuille flow is written as

$$v_1 = -\frac{h^2}{2\mu}\frac{dP}{dx_1}\frac{x_2}{h}(1-\frac{x_2}{h}) + \frac{Ux_2}{h}\ .$$

The shear stress is

$$\sigma_{12} = \mu\frac{dv_1}{dx_2} = -\frac{h}{2}\frac{dP}{dx_1}(1-\frac{2x_2}{h}) + \frac{\mu U}{h}\ .$$

Finally, the flow rate is

$$Q = \int_0^h v_1\,dx_2 = -\frac{h^3}{12\mu}\frac{dP}{dx_1} + \frac{Uh}{2}\ .$$

Solution 8.3

From the geometrical point of view, this flow occurs between two concentric cylinders as in the circular Couette flow. The inner cylinder of radius R_1 and the outer one of radius R_2 have a rate of angular rotation ω_1 and ω_2, respectively. The viscous fluid between the cylinders is also subject to an axial pressure gradient. As the flow is in steady state ($\partial/\partial t = 0$) and presents a symmetry of revolution ($\partial/\partial\theta = 0$), the velocity profile depends only on r. One has

$$v_r = v_r(r),\ v_\theta = v_\theta(r),\ v_z = v_z(r),\ p = p(r, z)\ . \tag{8.1}$$

As the fluid sticks to the wall, the boundary conditions are

$$\begin{aligned} &v_r(R_1) = v_r(R_2) = 0, \qquad v_\theta(R_1) = \omega_1 R_1,\ v_\theta(R_2) = \omega_2 R_2, \\ &v_z(R_1) = v_z(R_2) = 0\ . \end{aligned} \tag{8.2}$$

By a similar reasoning as the one for the circular Couette flow, it is possible to show that the component v_r vanishes everywhere (cf. (**B**8.98) and (**B**8.99)). The Navier-Stokes equations in cylindrical coordinates (A.32)-(A.34) become

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{v_\theta^2}{r}\ , \tag{8.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta^2}{r} = 0\ , \tag{8.4}$$

$$-\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right) = 0. \tag{8.5}$$

The Couette solution (**B**8.102) remains valid

$$v_\theta(r) = Ar + \frac{B}{r} = \frac{\omega_2 R_2^2 - \omega_1 R_1^2}{R_2^2 - R_1^2}\, r - \frac{(\omega_2 - \omega_1)R_1^2 R_2^2}{R_2^2 - R_1^2}\,\frac{1}{r}\ . \tag{8.6}$$

Relation (8.3) gives

$$p = \rho\int_{R_1}^{r} \frac{v_\theta^2}{r'}dr' + f(z)\ , \tag{8.7}$$

where v_θ is the Couette solution and $f(z)$ is an undetermined function of z. Introducing (8.7) in (8.5), one finds

$$-\frac{df}{dz} + \mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = 0\ . \tag{8.8}$$

As f does depend only on z and v_z is only function of r, one has

$$\frac{df}{dz} = \mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -C\ ,$$

where C is a constant. Here we will refer to the development described in pages **B**295 and **B**296 for the integration of v_z. The solutions are written by taking the boundary conditions (8.2) into account

$$f(z) = -Cz + D \; , \tag{8.9}$$

$$v_z(r) = \frac{C}{4\mu}\left[-r^2 + \frac{R_2^2 - R_1^2}{\ln(R_2/R_1)}\ln r + \frac{R_1^2\ln R_2 - R_2^2\ln R_1}{\ln(R_2/R_1)}\right] . \tag{8.10}$$

The factor D is a constant. The pressure field is given by

$$p(r,z) = \rho\int_{R_1}^{r}\frac{v_\theta^2}{r'}dr' - Cz + D \; . \tag{8.11}$$

Pressure is known up to a constant D, that will set the reference pressure; the pressure gradient $-C$ acts in the direction of the axis and finally, the first term of the right hand side of (8.11) balances the centrifugal force of the rotating fluid. Note that the axial velocity does not depend on the angular velocity of the cylinders, while the azimuthal velocity v_θ is independent of the pressure gradient.

Solution 8.4

We will refer to the spherical coordinates (r,θ,φ) as in figure **B**8.20. The rotation axis of the sphere with the angular velocity $\boldsymbol{\Omega} = \Omega\boldsymbol{e}_{x_3}$ is axis x_3. As a consequence of the problem's symmetries, the velocity field has only a single component such that

$$\boldsymbol{v} = v_\varphi(r,\theta)\boldsymbol{e}_\varphi \; . \tag{8.12}$$

We solve the Stokes equations with the boundary conditions

$$\boldsymbol{v} = 0 \quad \text{in} \quad r = \infty \tag{8.13}$$

$$v_\varphi = \Omega R\sin\theta \quad \text{in} \quad r = R \; . \tag{8.14}$$

The form of the boundary conditions (8.14) suggest to search the solution under the form

$$v_\varphi = \Omega R f(r)\sin\theta \; , \quad p = p_\infty \; . \tag{8.15}$$

One verifies that the mass conservation equation (**B**B.30) is trivially verified by (8.15). The pressure gradient does not intervene in (**B**B.33) because of axial symmetry ($\partial/\partial\varphi = 0$). One has

$$\Delta v_\varphi - \frac{v_\varphi}{r^2\sin^2\theta} = \Omega R\sin\theta\left(f'' + \frac{2f'}{r} - \frac{2f}{r^2}\right) = 0 \; . \tag{8.16}$$

The f solution is written as $f(r) = \sum_{n=-\infty}^{+\infty} C_n r^n$. This gives

$$f(r) = C_1 r + \frac{C_2}{r^2} \ . \tag{8.17}$$

The boundary conditions (8.13) and (8.14) impose $C_1 = 0$ and $C_2 = R^2$, respectively. The velocity field around the rotating sphere is

$$v_\varphi = \Omega \frac{R^3}{r^2} \sin\theta \, \boldsymbol{e}_\varphi \ .$$

Solution 8.5

The boundary conditions are

$$v_\varphi = \Omega_1 R_1 \sin\theta \quad \text{in} \quad r = R_1 \tag{8.18}$$

$$v_\varphi = \Omega_2 R_2 \sin\theta \quad \text{in} \quad r = R_2 \ . \tag{8.19}$$

The considerations of the previous exercise remain valid for the velocity profile search under the form (8.15)

$$v_\varphi = f(r)\sin\theta \ , \quad p = p_\infty \ . \tag{8.20}$$

We will note that we do not use anymore the factor ΩR, as now we have two radii and two angular velocities to take care of. The equation to solve is thus

$$\Delta v_\varphi - \frac{v_\varphi}{r^2 \sin^2\theta} = \sin\theta \left(f'' + \frac{2f'}{r} - \frac{2f}{r^2} \right) = 0 \ , \tag{8.21}$$

whose solution is written as

$$f(r) = C_1 r + \frac{C_2}{r^2} \ . \tag{8.22}$$

The imposition of the boundary conditions (8.18) and (8.19) yields

$$C_1 = \frac{\Omega_2 R_2^3 - \Omega_1 R_1^3}{R_2^3 - R_1^3}, \quad C_2 = \frac{\Omega_1 - \Omega_2}{R_2^3 - R_1^3} R_1^3 R_2^3 \ . \tag{8.23}$$

Solution 8.6

We consider the streamline SA from the free surface S toward the orifice Or of the enclosure (cf. figure 8.1) and we apply to it the Bernoulli theorem (**B**8.223) to obtain

$$p_S + \frac{\rho v_S^2}{2} + \rho\chi_S = p_A + \frac{\rho v_A^2}{2} + \rho\chi_A \ .$$

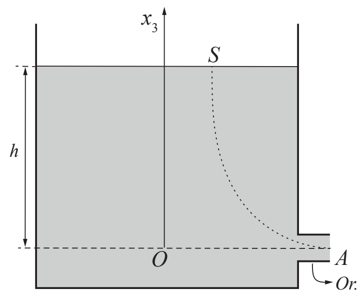

Figure 8.1 Enclosure with free surface and orifice.

By (**B**8.193), one obtains

$$-g = -\frac{\partial \chi}{\partial x_3} \ ,$$

and thus $\chi = gx_3 + C$. At the free surface the pressure is that of ambiant air ; it is the same situation at the orifice. Therefore $p_S = p_A = p_{air}$. If we set the origin of the x_3 axis at the level of the orifice, the contribution of $\rho\chi$ is equal to C. For the sake of simplicity, we set $C = 0$, while at the free surface $x_3 = h$, $\rho\chi_S = \rho gh$. On the free surface, the velocity v_S is zero (this is especially true when the enclosure is large) and setting $v_A = v$ one has

$$\rho gh = \frac{\rho}{2} v^2 \ . \tag{8.24}$$

This gives the sought relation, which is known as Torricelli formula.

Solution 8.7

From the Navier-Stokes equations (**B**8.17) without body forces and assuming a velocity field of the form

$$v_1 = v_1(x_2, x_3), \quad v_2 = v_3 = 0 \ , \tag{8.25}$$

the only relation that gives a non zero contribution is the one related to the v_1 velocity component. One finds

$$0 = -\frac{\partial p}{\partial x_1} + \mu \left(\frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2} \right) \ , \tag{8.26}$$

or

$$\frac{1}{\mu}\frac{\partial p}{\partial x_1} = \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2} = C \ , \tag{8.27}$$

as $p = p(x_1)$ and $v_1 = v_1(x_2, x_3)$.

On the elliptical wall $v_1 = 0$ and thus $A + B = 0$, or $B = -A$. We calculate the second order derivatives of the velocity that are injected in (8.27). One has

$$2A\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \frac{1}{\mu}\frac{\partial p}{\partial x_1} = C \ ,$$

From which we get A and $B = -A$. One obtains a generic solution, that will become particular for a given pressure gradient.

Solution 8.8

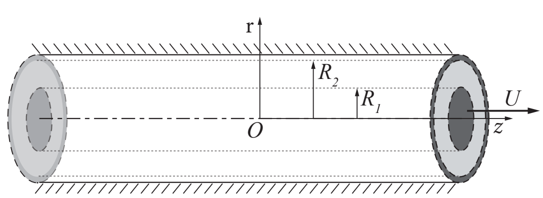

Figure 8.2 Flow between two concentric cylinders, one fixed and the other moving with the velocity U.

We work in cylindrical coordinates with the z axis in the direction of the axes of both cylinders (cf. figure 8.2). The only non zero velocity component is clearly v_z. Moreover $v_z = v_z(r)$.

The flow is kinematically forced by the displacement of the inner cylinder. No pressure gradient is involved in the fluid motion.

Equation (A.34) gives

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = 0 \ . \tag{8.28}$$

Integrating (8.28), one finds

$$v_z = C_1 \ln r + C_2 \ . \tag{8.29}$$

The boundary conditions are

$$v_z(r = R_1) = U \tag{8.30}$$

$$v_z(r = R_2) = 0 \; . \tag{8.31}$$

Imposing (8.30) and (8.31) to (8.29), one obtains the velocity field

$$v_z = \frac{U}{\ln \frac{R_1}{R_2}} \ln \frac{r}{R_2} \; .$$

The only non zero component of the stress tensor is σ_{rz} equal to

$$\sigma_{rz} = \mu \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) = \mu \frac{U}{\ln \frac{R_1}{R_2}} \frac{1}{r} \; . \tag{8.32}$$

The friction force per unit length that acts on the moving cylinder is given by the integral

$$\int_0^1 \sigma_{rz}|_{r=R1} 2\pi R_1 dz = 2\pi\mu \frac{U}{\ln \frac{R_1}{R_2}} \; .$$